2.3 #1 Consider the solution \( u(x,t) = 1 - x^2 - 2kt \) of the diffusion equation. Find the locations of its maximum and minimum in the closed rectangle \( \{ 0 \leq x \leq 1, 0 \leq t \leq T \} \)

Notice that \( u(x,t) \) decreases linearly in \( t \) and so is largest when \( t = 0 \). Also, \( u(x,0) \) is quadratic in \( x \) and so has a maximum at the vertex of \( u(x,0) = 1 - x^2 \). That is, the maximum of \( u(x,t) \) is 1 at \( x = 0 \).

2.3 #2 Consider a solution of

\[
\begin{align*}
  u_t &= u_{xx} \quad \text{in } \{ 0 \leq x \leq \ell, 0 \leq t \leq T \}.
\end{align*}
\]

(a) Let \( M(T) = \max[u(x,t)|0 \leq x \leq \ell, 0 \leq t \leq T] \). Does \( M(T) \) increase or decrease as a function of \( T \)?

(b) Let \( m(T) = \min[u(x,t)|0 \leq x \leq \ell, 0 \leq t \leq T] \). Does \( m(T) \) increase or decrease as a function of \( T \)?

Let \( T_1 < T_2 \). Then, \( \{u(x,t)|0 \leq x \leq \ell, 0 \leq t \leq T_1 \} \subset \{u(x,t)|0 \leq x \leq \ell, 0 \leq t \leq T_2 \} \). Thus, the maximum over these sets must not decrease and the minimum must not increase. So, \( M(T) \) is an increasing function and \( m(T) \) is a decreasing function. \( \square \)

2.3 #4 Consider the equation \( u_t = u_{xx} \) in \( \{ 0 < x < 1, t > 0 \} \) with \( u(0,t) = u(1,t) = 0 \) and \( u(x,0) = 4x(1-x) \).

(a) Show 0 < \( u(x,t) < 1 \) for all \( t > 0 \) and 0 < \( x < 1 \).

\[ \text{proof:} \] Using the strong maximum principle, we know that both the minimum and the maximum of \( u \) cannot happen in the interior of the rectangle unless \( u \) is constant. Since \( u \) is not constant along \( t = 0 \), \( u \) is not constant at all. Along the boundary of the rectangle \( \{ 0 < x < 1, t > 0 \} \) \( u \) attains a minimum of 0 and a maximum of 1. Thus 0 < \( u(x,t) < 1 \) on \( \{ 0 < x < 1, t > 0 \} \). \( \square \)

(b) Show that \( u(x,t) = u(1-x,t) \) for all \( t \geq 0 \) and 0 ≤ \( x \leq 1 \).

\[ \text{proof:} \] Notice that if \( v(x,t) = u(1-x,t) \) then \( v \) satisfies the diffusion equation (since \( v_t(x,t) = u_t(1-x,t) \) and \( v_{xx}(x,t) = u_{xx}(1-x,t) \)) with the conditions \( v(0,t) = v(1,t) = 0 \) and \( v(x,0) = 4x(1-x) \). But since the solution to the diffusion equation is unique, we know that \( u = v \), that is, \( u(x,t) = u(1-x,t) \). \( \square \)

(c) Use the energy method to show that \( \int_0^1 u^2 \, dx \) is a strictly decreasing function of \( t \).

\[ \text{proof:} \] Let \( E(t) = \int_0^1 u^2 \, dx \). Then

\[
E'(t) = \frac{d}{dt} \int_0^1 u^2 \, dx = \int_0^1 \frac{d}{dt} u^2 \, dx = 2 \int_0^1 uu_t \, dx = 2 \int_0^1 uu_{xx} \, dx = 2uu_x|_0^1 - 2 \int_0^1 (u_x)^2 \, dx \\
= -2 \int_0^1 (u_x)^2 \, dx \leq 0
\]

If \( -2 \int_0^1 (u_x)^2 \, dx = 0 \) then \( u_x = 0 \). That is, \( u \) is constant, but we know that \( u(0,t) = 0 \) but \( u > 0 \) inside the rectangle, so \( u \) is not constant. So, \( E'(t) < 0 \), that is, \( E(t) \) is strictly decreasing. \( \square \)

2.3 #5 Show that the maximum principle does not hold for

\[
\begin{equation}
  u_t = xu_{xx}
\end{equation}
\]

by:
(a) Verify that \( u(x, t) = -2xt - x^2 \) is a solution. Find the location of its maximum in \([-2 \leq x \leq 2, 0 \leq t \leq 1]\).

**Proof:** First, we compute derivatives and show that \( u \) does satisfy (1).

\[
\begin{align*}
  u_t &= -2x \quad \text{and} \quad u_{xx} = -2 \\
  \Rightarrow 
  u_t &= -2x = xu_{xx}.
\end{align*}
\]

To find the maximum, we find where the gradient is zero and check the boundaries of the rectangle. It turns out that the gradient \( \nabla u = 0 \) when \( (x, t) = (0, 0) \) and \( u(0, 0) = 0 \). We know that when \( x = -2 \), \( u(-2, t) = 4t - 4 \) which has a maximum of 0 at \( t = 1 \). When \( x = 2 \), \( u(2, t) = -4t - 4 \) which has a maximum of \( -4 \) when \( t = 0 \). When \( t = 0 \), \( u(x, 0) = -x^2 \) which has a maximum of 0 at \( x = 0 \). And, when \( t = 1 \), \( u(x, 1) = -2x - x^2 \) which has a maximum of 1 when \( x = -1 \). Thus, \( u \) attains a maximum of 1 at \((-1, 1)\). The maximum principle says that the maximum must happen on the boundary, but for the diffusion equation, it should not happen at the top of the rectangle (which is what happened). \( \square \)

(b) Where does the proof of the maximum principle break down for this equation?

In the proof of the maximum principle, we let \( v = u(x, t) + \epsilon x^2 \) and we see that

\[
v_t - kxv_{xx} = u_t - kxu_{xx} - 2xk\epsilon = -2x\epsilon.
\]

But \(-2x\epsilon\) is not necessarily less than 0 as is needed in the maximum principle proof.

### 2.3 #6 Prove the comparison principle: If \( u \) and \( v \) are two solutions, and if \( u \leq v \) for \( t = 0, x = 0, \) and \( x = \ell \), then \( u \leq v \) for \( t \geq 0, 0 \leq x \leq \ell \).

**Proof:** Let \( u \) and \( v \) both satisfying \( u_t = u_{xx} \) be such that \( u \leq v \) for \( t = 0, x = 0, \) and \( x = \ell \). Define \( w = u - v \). Then \( w_t = u_t - v_t = u_{xx} - v_{xx} = w_{xx} \). Also, we know that \( w(x, t) \leq 0 \) when \( t = 0, x = 0, \) or \( x = \ell \). So by the maximal principle, \( w \leq 0 \) when \( t \geq 0, 0 \leq x \leq \ell \). \( \square \)

### 2.3 #7

(a) More generally, if

\[
\begin{align*}
  u_t - ku_{xx} &= f, \\
  v_t - ku_{xx} &= g, \\
  f &\leq g, \\
  u &\leq v \text{ at } x = 0, x = \ell, \text{ and } t = 0
\end{align*}
\]

then \( u \leq v \) for \( 0 \leq x \leq \ell, t \geq 0 \).

**Proof:** Suppose \( u \) and \( v \) satisfy (2)-(5) and let \( w = u - v \). Then \( w_t - kw_{xx} = f - g \leq 0 \) on \( 0 \leq x \leq \ell, t \geq 0 \). Let \( w^\epsilon = w + \epsilon x^2 \), for some \( \epsilon > 0 \), and suppose that \( w \) has an interior max at \( (x_0, t_0) \in \{0 < x < \ell, t > 0\} \). Then \( w_t(x_0, t_0) = 0 \) and \( w_{xx}(x_0, t_0) \leq 0 \). Thus, for \( \epsilon \) small enough

\[
w_t^\epsilon(x_0, t_0) - kw_{xx}^\epsilon(x_0, t_0) = -kw_{xx}(x_0, t_0) - 2k\epsilon > 0.
\]

This is true for all \( \epsilon > 0 \). Thus

\[
w_t(x_0, t_0) - kw_{xx}(x_0, t_0) < 0.
\]

But this contradicts that \( w_t - kw_{xx} \leq 0 \). Thus, \( w \) does not attain a maximum in the interior. That is, the maximum of \( w \) is on the boundary. But, \( w \leq 0 \) on the boundary so \( w \leq 0 \) in \( \{0 \leq x \leq \ell, t \geq 0\} \).

(b) If \( v_t - v_{xx} \geq \sin x \) for \( 0 \leq x \leq \pi, t > 0 \), and if \( v(0, t) \geq 0, v(\pi, t) \geq 0 \) and \( v(x, 0) \geq \sin x \), show that \( v(x, t) \geq (1 - e^{-t}) \sin x \).

**Proof:** Let \( u(x, t) = (1 - e^{-t}) \sin x \). Then \( u_t - u_{xx} = e^{-t} \sin x + (1 - e^{-t}) \sin x = \sin x \). Suppose also \( v_t - v_{xx} \geq \sin x \), that is, suppose \( v_t - v_{xx} = f \), where \( f \leq \sin x \). Also assume \( v(0, t) \geq 0 = u(0, t) \), \( v(\pi, t) \geq 0 = u(\pi, t) \), and \( v(x, 0) \geq \sin x \geq 0 = u(x, 0) \). Then by part (a), we have that \( v(x, t) \geq u(x, t) \) when \( 0 \leq x \leq \pi \) and \( t > 0 \). That is, \( v(x, t) \geq (1 - e^{-t}) \sin x \) when \( 0 \leq x \leq \pi \) and \( t > 0 \).