Solve \( u_{tt} = c^2 u_{xx} \), \( u(x, 0) = \log(1 + x^2) \), \( u_t(x, 0) = 4 + x \).

**Solution** We know that the wave equation has a solution in the form
\[
    u(x, t) = \frac{1}{2} \left[ \phi(x + ct) + \phi(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds
\]

The initial data says
\[
    \phi(x) = u(x, 0) = \log(1 + x^2) \\
    \psi(x) = u_t(x, 0) = 4 + x.
\]

Thus,
\[
    u(x, t) = \frac{1}{2} \left[ \log(1 + (x + ct)^2) + \log(1 + (x - ct)^2) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} 4 + s \, ds
\]
\[
    = \frac{1}{2} \left[ \log(1 + (x + ct)^2) + \log(1 + (x - ct)^2) \right] + \frac{1}{2c} \left[ 8ct + \frac{1}{2} (x + ct)^2 - \frac{1}{2} (x - ct)^2 \right]
\]

The hammer blow problem. **Solution** Notice that in this problem,
\[
    u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds,
\]
where
\[
    \psi(x) = \begin{cases} 
    1 & \text{if } |x| < a \\
    0 & \text{if } |x| \geq a
\end{cases}
\]

First, we look at all 6 cases for the location of \( x + ct \) and \( x - ct \) with respect to \(-a\) and \(a\). These are organized in the following table:

<table>
<thead>
<tr>
<th>Case</th>
<th>Interval</th>
<th>( u(x, t) )</th>
<th>Picture</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x - ct &lt; x + ct &lt; -a &lt; a )</td>
<td>( u(x, t) = \int_{x-ct}^{x+ct} 0 , ds = 0 )</td>
<td>![Picture 1]</td>
</tr>
<tr>
<td>2</td>
<td>(-a &lt; a &lt; x - ct &lt; x + ct )</td>
<td>( u(x, t) = \int_{x-ct}^{x+ct} 0 , ds = 0 )</td>
<td>![Picture 2]</td>
</tr>
<tr>
<td>3</td>
<td>(-a &lt; x - ct &lt; x + ct &lt; a )</td>
<td>( u(x, t) = \int_{x-ct}^{x+ct} 1 , ds = 2ct )</td>
<td>![Picture 3]</td>
</tr>
<tr>
<td>4</td>
<td>(-a &lt; x - ct &lt; a &lt; x + ct )</td>
<td>( u(x, t) = \int_{x-ct}^{a} 1 , ds = a - x + ct )</td>
<td>![Picture 4]</td>
</tr>
<tr>
<td>5</td>
<td>( x - ct &lt; -a &lt; x + ct &lt; a )</td>
<td>( u(x, t) = \int_{-a}^{x+ct} 1 , ds = x + ct + a )</td>
<td>![Picture 5]</td>
</tr>
<tr>
<td>6</td>
<td>( x - ct &lt; -a &lt; a &lt; x + ct )</td>
<td>( u(x, t) = \int_{-a}^{a} 1 , ds = 2a )</td>
<td>![Picture 6]</td>
</tr>
</tbody>
</table>
So now we just consider this for each of the given \( t \) values:

\[
t = \frac{a}{2x}
\]

<table>
<thead>
<tr>
<th>Case</th>
<th>Interval</th>
<th>( u(x, t) = \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} 0 , ds = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x &lt; -\frac{3a}{2} )</td>
<td>( u(x, t) = \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} 0 , ds = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{3a}{2} &lt; x )</td>
<td>( u(x, t) = \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} 0 , ds = 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( -\frac{a}{2} &lt; x &lt; \frac{a}{2} )</td>
<td>( u(x, t) = \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} 1 , ds = a )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{a}{2} &lt; x &lt; \frac{3a}{2} )</td>
<td>( u(x, t) = \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} 1 , ds = \frac{3a}{2} - x )</td>
</tr>
<tr>
<td>5</td>
<td>( -\frac{3a}{2} &lt; x &lt; -\frac{a}{2} )</td>
<td>( u(x, t) = \int_{-a}^{x+\frac{a}{2}} 1 , ds = x + \frac{3a}{2} )</td>
</tr>
<tr>
<td>6</td>
<td>( x &lt; -\frac{a}{2}, x &gt; \frac{a}{2} )</td>
<td>( \rightarrow )</td>
</tr>
</tbody>
</table>

Now we plot \( u \) on these intervals to get
\( t = \frac{a}{c} \)

<table>
<thead>
<tr>
<th>Case</th>
<th>Interval</th>
<th>( u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x &lt; -2a )</td>
<td>( u(x, t) = \int_{x-a}^{x+a} 0 , ds = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( -2a &lt; x )</td>
<td>( u(x, t) = \int_{x-a}^{x+a} 0 , ds = 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( 0 &lt; x &lt; 0 )</td>
<td>( u(x, t) = \int_{x-a}^{x+a} 0 , ds = 0 )</td>
</tr>
<tr>
<td>4</td>
<td>( 0 &lt; x &lt; 2a )</td>
<td>( u(x, t) = \int_{x-a}^{x+a} 1 , ds = 2a - x )</td>
</tr>
<tr>
<td>5</td>
<td>( -2a &lt; x &lt; 0 )</td>
<td>( u(x, t) = \int_{x+a}^{x+2a} 1 , ds = x + 2a )</td>
</tr>
<tr>
<td>6</td>
<td>( x &lt; 0, 0 &lt; x )</td>
<td>( u(x, t) = \int_{-2a}^{2a} 1 , ds = 2a )</td>
</tr>
</tbody>
</table>

\[ t = \frac{3a}{2c} \]

<table>
<thead>
<tr>
<th>Case</th>
<th>Interval</th>
<th>( u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x &lt; -\frac{5a}{2} )</td>
<td>( u(x, t) = \int_{x-\frac{5a}{2}}^{x+\frac{5a}{2}} 0 , ds = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( -\frac{5a}{2} &lt; x )</td>
<td>( u(x, t) = \int_{x-\frac{5a}{2}}^{x+\frac{5a}{2}} 0 , ds = 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{a}{2}, x &lt; -\frac{a}{2} )</td>
<td>( u(x, t) = \int_{x-\frac{5a}{2}}^{x+\frac{5a}{2}} 0 , ds = 0 )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{a}{2} &lt; x &lt; \frac{5a}{2} )</td>
<td>( u(x, t) = \int_{x-\frac{5a}{2}}^{x+\frac{5a}{2}} 1 , ds = -x + \frac{5a}{2} )</td>
</tr>
<tr>
<td>5</td>
<td>( -\frac{5a}{2} &lt; x &lt; -\frac{a}{2} )</td>
<td>( u(x, t) = \int_{x-\frac{5a}{2}}^{x+\frac{5a}{2}} 1 , ds = x + \frac{5a}{2} )</td>
</tr>
<tr>
<td>6</td>
<td>( -\frac{a}{2} &lt; x &lt; \frac{a}{2} )</td>
<td>( u(x, t) = \int_{-a}^{a} 1 , ds = 2a )</td>
</tr>
</tbody>
</table>
\[ t = \frac{2a}{c} \]

<table>
<thead>
<tr>
<th>Case</th>
<th>Interval</th>
<th>( u(x,t) = \int_{x-2a}^{x+2a} 0 , ds = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x &lt; -3a )</td>
<td>( u(x,t) = \int_{x-2a}^{x+2a} 0 , ds = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>(-3a &lt; x )</td>
<td>( u(x,t) = \int_{x-2a}^{x+2a} 0 , ds = 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( a &lt; x, x &lt; -a \rightarrow\leftarrow )</td>
<td>( u(x,t) = \int_{x-2a}^{x+2a} 1 , ds = 4a )</td>
</tr>
<tr>
<td>4</td>
<td>( a &lt; x &lt; 3a )</td>
<td>( u(x,t) = \int_{x-2a}^{a} 1 , ds = 3a - x )</td>
</tr>
<tr>
<td>5</td>
<td>(-3a &lt; x &lt; -a )</td>
<td>( u(x,t) = \int_{-a}^{x+2a} 1 , ds = x + 3a )</td>
</tr>
<tr>
<td>6</td>
<td>(-a &lt; x &lt; a )</td>
<td>( u(x,t) = \int_{-a}^{a} 1 , ds = 2a )</td>
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</tbody>
</table>

\[ t = \frac{5a}{c} \]
First, we can rewrite the equation as

\[ D^2 u - 3 D T u - 4 T^2 u = 0, \]

where \( D = \frac{\partial}{\partial x} \) and \( T = \frac{\partial}{\partial t} \). Factoring gives

\[ (D + T)(D - 4T)u = 0. \]

Now, let’s set \( v = (D - 4T)u \). This gives the system of PDEs:

\[ u_x - 4u_t = v \]
\[ v_x + v_t = 0. \]
To solve this system, we solve the second equation for \(v\) and then solve the first with \(v\) plugged back in. To solve the second equation, we recognize that it can be written as a dot product:

\[
\nabla v \cdot \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = 0.
\]

Thus, \(\nabla v\) is perpendicular to \(\left( \begin{array}{c} 1 \\ 1 \end{array} \right)\). Thus, \(v\) is constant on lines parallel to \(\left( \begin{array}{c} 1 \\ 1 \end{array} \right)\). That is, \(v\) is constant on lines of the form \(x - t = c\). So \(v = f(x - t)\). Putting this into the first equation in our system gives

\[
u_x - 4u_t = f(x - t).
\]

Now we can do a change of coordinates to solve this equation.

Let \(\tilde{x} = x - 4t\) and \(\tilde{t} = 4x + t\).

Doing this change of coordinates gives us that

\[
u_x = u_{\tilde{x}} + 4u_{\tilde{t}}
\]

\[
u_t = -4u_{\tilde{x}} + u_{\tilde{t}}.
\]

Rewriting the equation in terms of \(\tilde{x}\) and \(\tilde{t}\), we have

\[
u_{\tilde{x}} + 4u_{\tilde{t}} + 4(-4u_{\tilde{x}} + u_{\tilde{t}}) = f(x - t)
\]

\[
-15u_{\tilde{x}} = f(x - t).
\]

Now, we just solve for \(u\) by integrating. This gives us

\[
-15u = \int f(x - t)d\tilde{x} + g(\tilde{t}).
\]

If we assume \(f\) has antiderivative \(cF\) (for some appropriate constant), we can write

\[
u(x, t) = F(x - t) + g(4x + t).
\]

Now, we use the initial conditions to find \(F\) and \(g\).

\[
u(x, 0) = x^2 \implies F(x) + g(4x) = x^2
\]

and

\[
u_t(x, 0) = e^x \implies -F'(x) + g'(4x) = e^x.
\]

If we differentiate the first equation, we get

\[
F'(x) + 4g'(4x) = 2x.
\]

Adding these two equations gives us

\[
5g'(4x) = e^x + 2x.
\]

Integrating gives

\[
\frac{5}{4}g(4x) = e^x + x^2 \implies g(x) = \frac{4}{5}e^{\frac{x}{4}} + \frac{1}{20}x^2.
\]

Now, we put this back into the first equation and we get

\[
F(x) = x^2 - g(4x) = x^2 - \frac{4}{5}(e^x + x^2) = \frac{1}{5}x^2 - \frac{4}{5}e^x.
\]

Thus,

\[
u(x, t) = F(x - t) + g(4x - t) = \frac{1}{5}(x - t)^2 - \frac{4}{5}e^{(x-t)} + \frac{4}{5}e^{\frac{4x-t}{4}} + \frac{1}{20}(4x + t)^2
\]
As in the previous problem, we rewrite the PDE using operators as
\[(D^2 + DT - 20T^2)u = 0,\]
where \(D, T\) are defined as before. Factoring gives us the equation
\[(D - 4T)(D + 5T)u = 0.\]
Now, as before, we substitute \(v = (D + 5T)u\) to get the system of equations
\[u_x + 5u_t = v\]
\[v_x - 4v_t = 0.\]
Again, we recognize the second equation as a dot product and rewrite it as
\[\nabla v \cdot \begin{pmatrix} 1 \\ -4 \end{pmatrix} = 0.\]
This tells us that since \(\nabla v\) is perpendicular to \(\begin{pmatrix} 1 \\ -4 \end{pmatrix}\), \(v\) is constant on lines of the form \(4x + t = c\). That is, \(v(x,t) = f(4x + t)\). Now, we plug this into the first equation to get the equation
\[u_x + 5u_t = f(4x + t).\]
We can use the change of coordinates
\[\tilde{x} = x + 5t, \tilde{t} = 5x - t\]
to get
\[u_x = u_{\tilde{x}} + 5u_t \text{ and } u_t = 5u_{\tilde{x}} - u_t.\]
This transforms our PDE to
\[u_x + 5u_t + 5(5u_{\tilde{x}} - u_t) = f(4x + t) \quad \Rightarrow \quad 6u_{\tilde{x}} = f(4x + t).\]
Integrating gives (again, assuming \(cF\) is the antiderivative of \(f\) with the appropriate constant \(c\))
\[u = F(4x + t) + g(\tilde{t}).\]
Thus, \(u(x,t) = F(4x + t) + g(5x - t)\).
Now, we use the initial conditions:
\[u(x,0) = \phi(x) \quad \Rightarrow \quad F(4x) + g(5x) = \phi(x)\]
and
\[u_t(x,0) = \psi(x) \quad \Rightarrow \quad F'(4x) - g'(5x) = \psi(x).\]
Differentiating the first gives

\[ 4F'(x) + 5g'(x) = \phi'(x). \]

Adding this to 5 times the second equation gives

\[ 9F'(4x) = \phi'(x) + 5\psi(x). \]

Thus,

\[ F'(x) = \frac{1}{9}\left(\phi'(\frac{x}{4}) + 5\psi(\frac{x}{4})\right). \]

Integrating gives

\[ F(x) = \frac{1}{36}\phi\left(\frac{x}{4}\right) + \frac{5}{36}\int_{0}^{x/4} \psi(s)\, ds. \]

Using the first equation again, we get

\[ g(5x) = \phi(x) - F(4x) = \phi(x) - \frac{1}{36}\phi(x) - \frac{5}{36}\int_{0}^{x/4} \psi(s)\, ds \]

Thus

\[ g(x) = \phi(x/5) - F(4x/5) = \phi(x/5) - \frac{1}{36}\phi(x/5) - \frac{5}{36}\int_{0}^{x/20} \psi(s)\, ds = \frac{35}{36}\phi(x) - \frac{5}{36}\int_{0}^{x/20} \psi(s)\, ds. \]

This gives us our solution:

\[ u(x,t) = \frac{1}{36}\phi\left(\frac{4x + t}{4}\right) + \frac{5}{36}\int_{0}^{4xt/4} \psi(s)\, ds + \frac{35}{36}\phi(5x - t) - \frac{5}{36}\int_{0}^{5xt/20} \psi(s)\, ds. \]

1. In a Vector Calculus class, Green’s Theorem is presented as:

\[ \oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_{D} \text{div} \mathbf{F}(x,y)\, dA, \tag{1} \]

where \( \mathbf{F} \) is a vector-valued function. Use this to prove these other versions of Green’s Theorem.

(a)

\[ \iint_{D} f \Delta g\, dA = \oint_{\partial D} f \nabla g \cdot \mathbf{n}\, ds - \iint_{D} \nabla f \cdot \nabla g\, dA. \]

To get this, we replace, in \([1]\) \( F = f\nabla g \) to get

\[ \oint_{\partial D} f \nabla g \cdot \mathbf{n}\, ds = \iint_{D} \text{div} (f\nabla g)\, dA. \]

Using the product rule, we compute

\[ \text{div}(f\nabla g) = \nabla f \cdot \nabla g + f\Delta g. \]

Thus, we have

\[ \oint_{\partial D} f \nabla g \cdot \mathbf{n}\, ds = \iint_{D} \nabla f \cdot \nabla g + f\Delta g\, dA. \]

Rearranging gives the desired result.
\[ \int \int_D (f \Delta g - g \Delta f) \, dA = \oint_{\partial D} (f \nabla g - g \nabla f) \cdot \mathbf{n} \, ds \]

For this one, we substitute in (1) \( F = f \nabla g - g \nabla f \) to get

\[ \oint_{\partial D} (f \nabla g - g \nabla f) \cdot \mathbf{n} \, ds = \int \int_D \text{div}(f \nabla g - g \nabla f) \, dA. \]

Using the product rule twice, we compute

\[ \text{div}(f \nabla g - g \nabla f) = \nabla \cdot \nabla f + f \Delta g - \nabla g \cdot \nabla f - f \Delta g = f \Delta g - g \Delta f. \]

Thus, we have

\[ \oint_{\partial D} (f \nabla g - g \nabla f) \cdot \mathbf{n} \, ds = \int \int_D f \Delta g - g \Delta f \, dA, \quad (2) \]

which is the desired result.