Gathering 4 Gardner
Celebration of Mind

Maps of Strange Worlds: Beyond the Four-Color Theorem

Dr. Susan Goldstine
St. Mary's College of Maryland

Map Coloring: A Brief History

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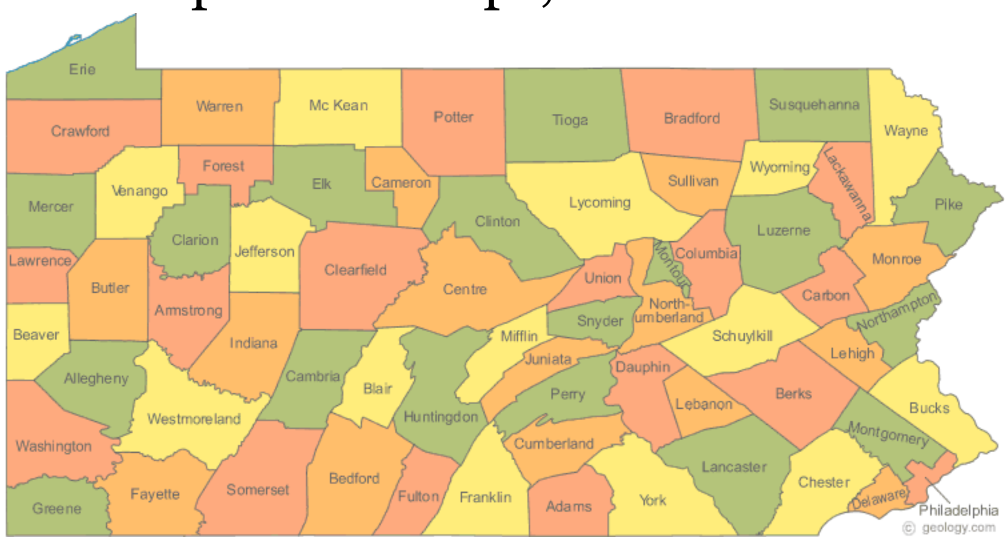

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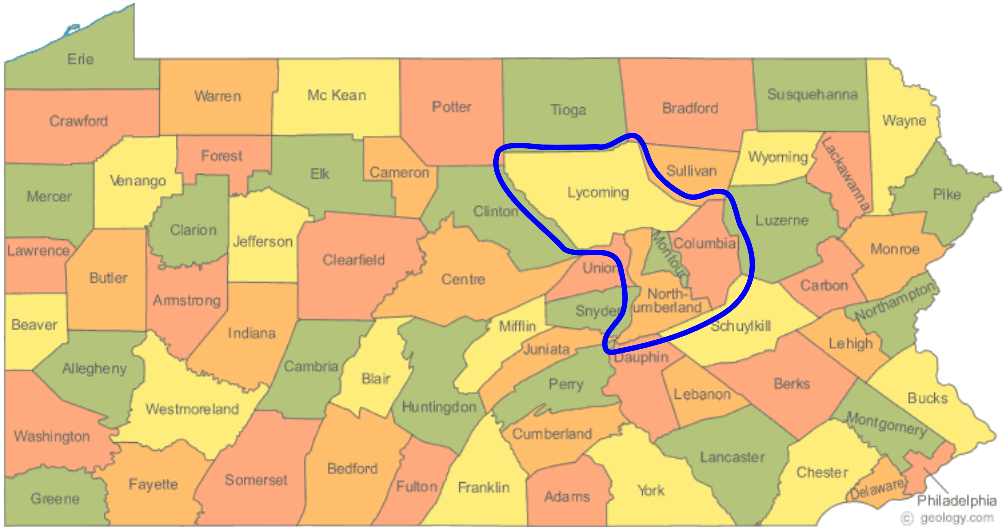

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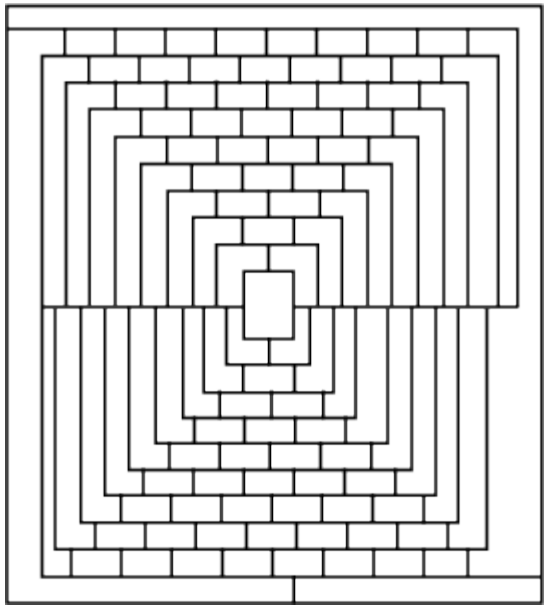

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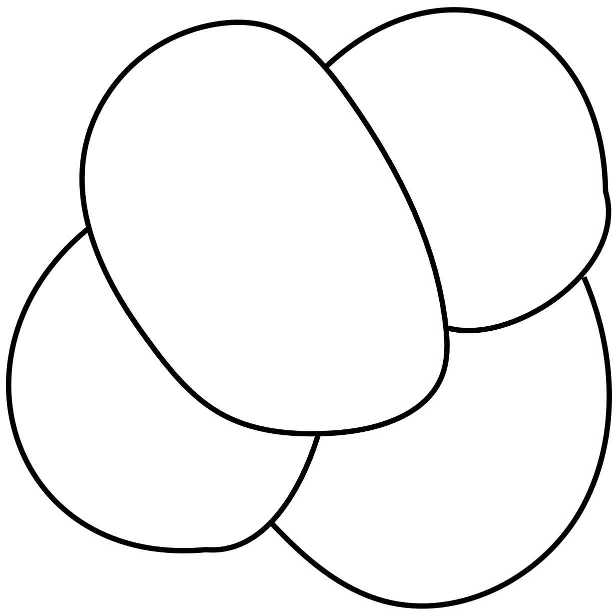
1996 Robertson, Sanders, Seymour, and Thomas give a quadratic-time algorithm for four-coloring a map, and a simplified proof

Understanding Maps

A map divides the plane into regions.



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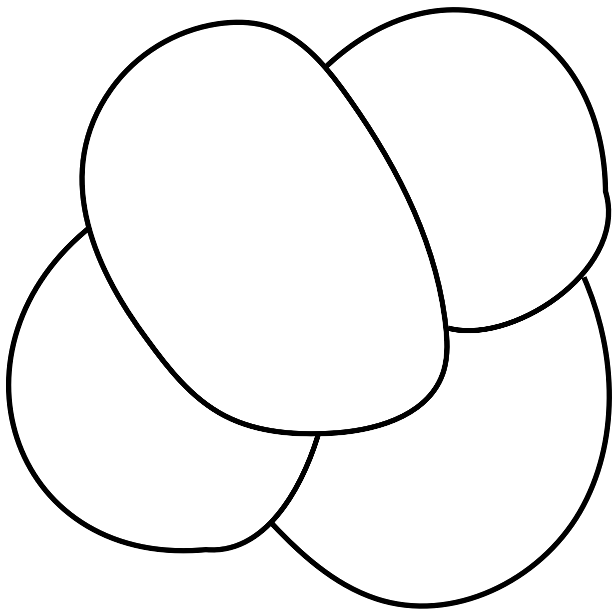
A map divides the plane/sphere into regions, a.k.a. faces.



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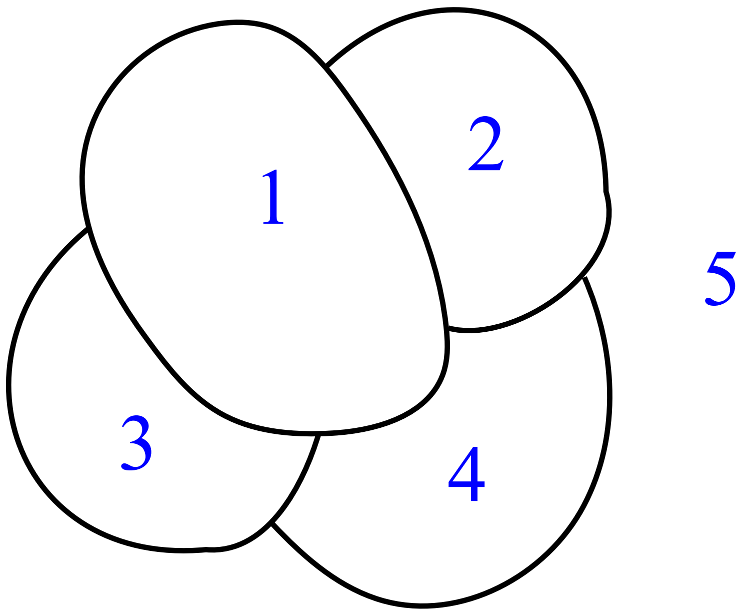

F = 5

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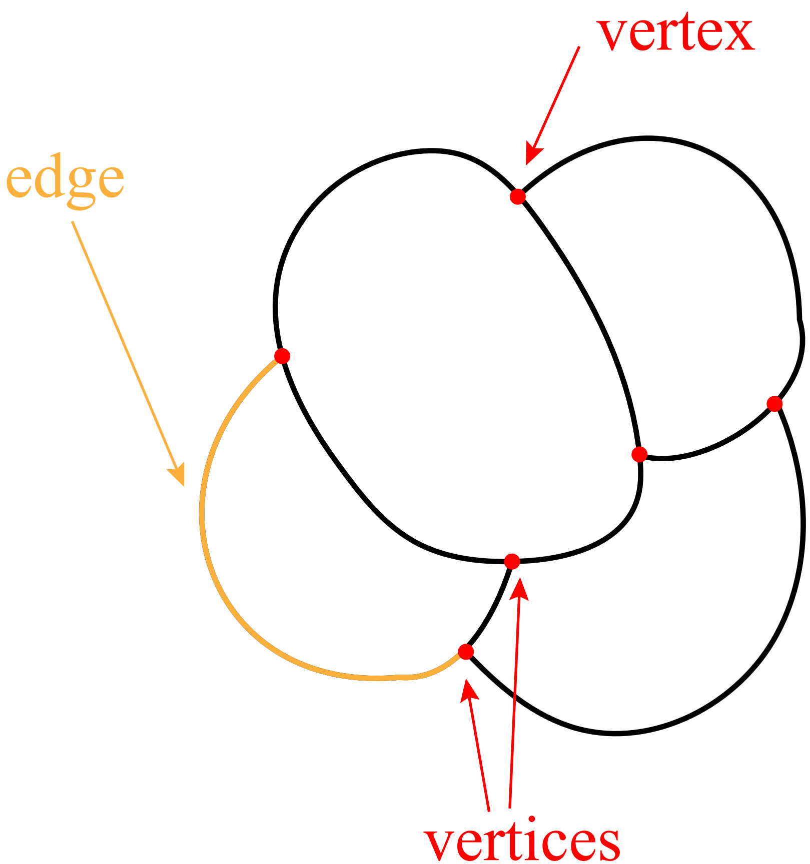

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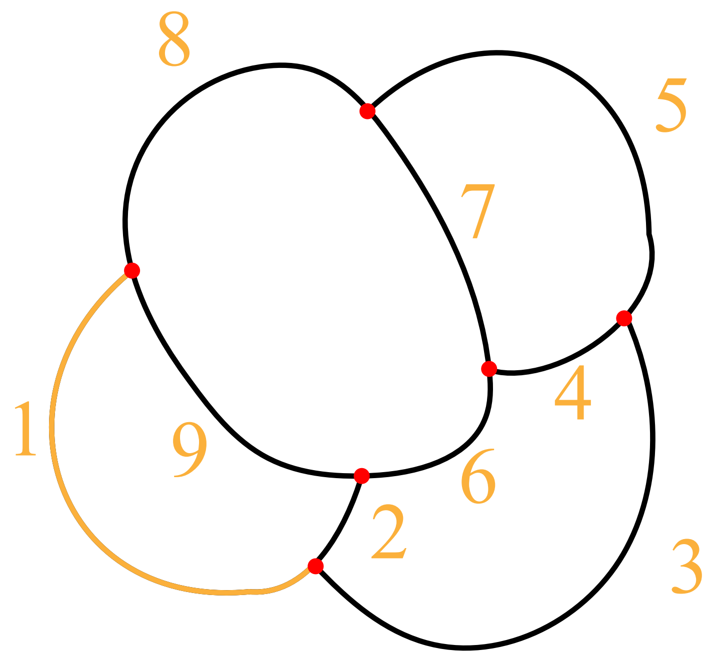

F = 5
E = 9

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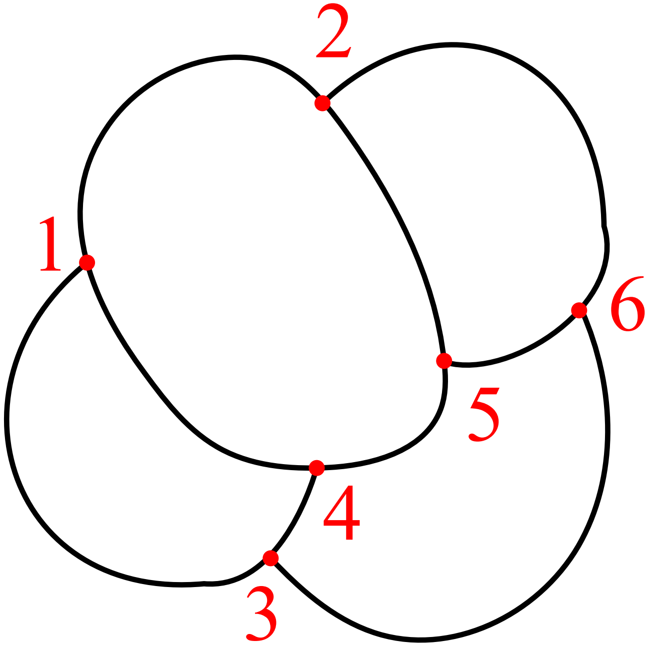

F = 5
E = 9
V = 6

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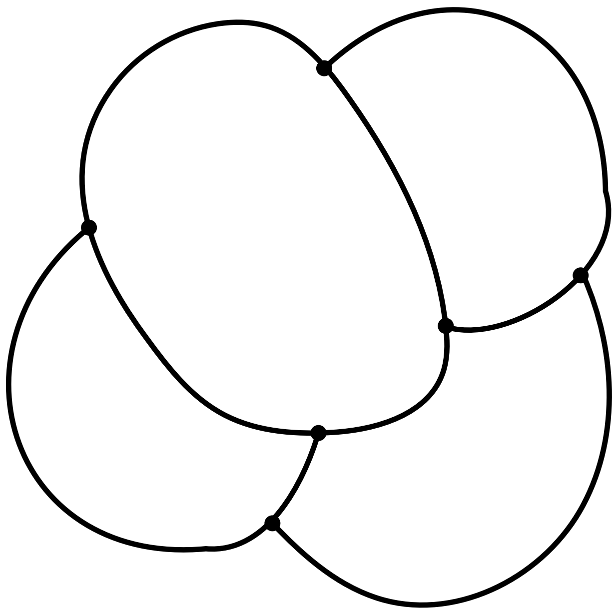


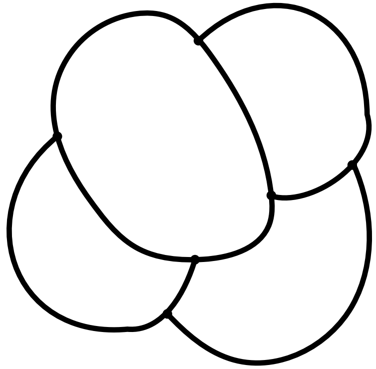
$F = 5$

$E = 9$

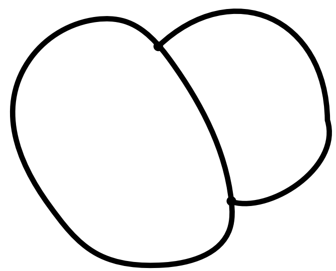
$V = 6$

$$V - E + F = 6 - 9 + 5 = 2$$

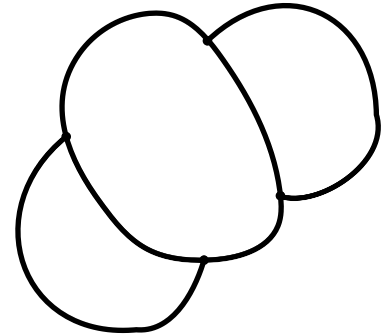
A Curious Pattern



$6 - 9 + 5 = 2$



$2 - 3 + 3 = 2$



$4 - 6 + 4 = 2$

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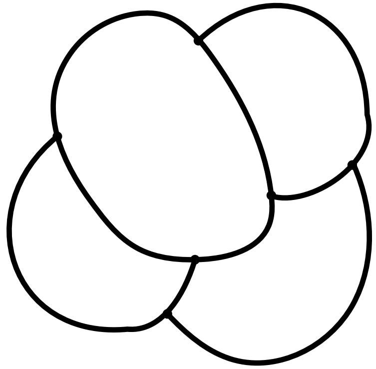

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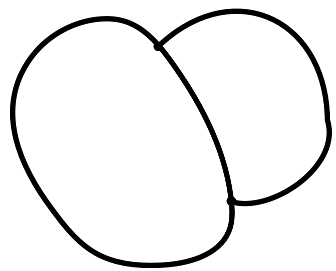

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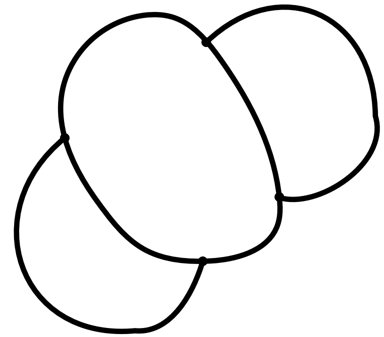
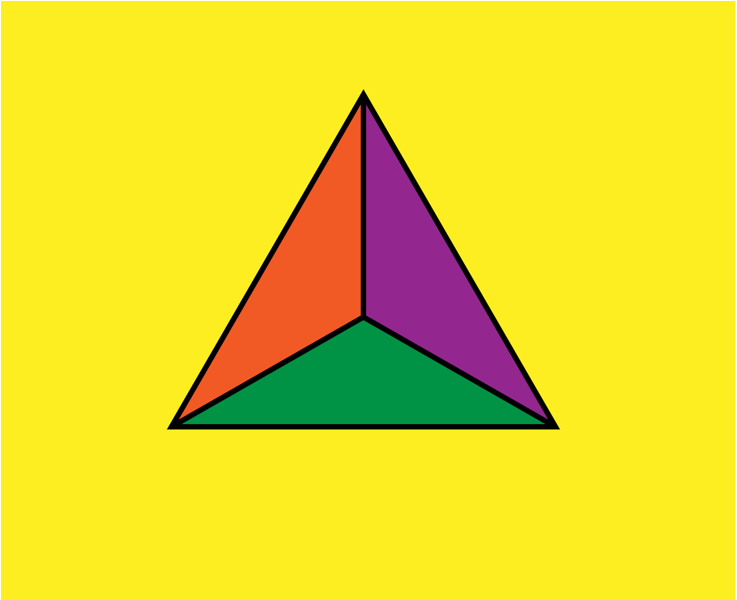


$4 - 6 + 4 = 2$

Euler's Formula: $V - E + F = 2$

Four Colors Needed



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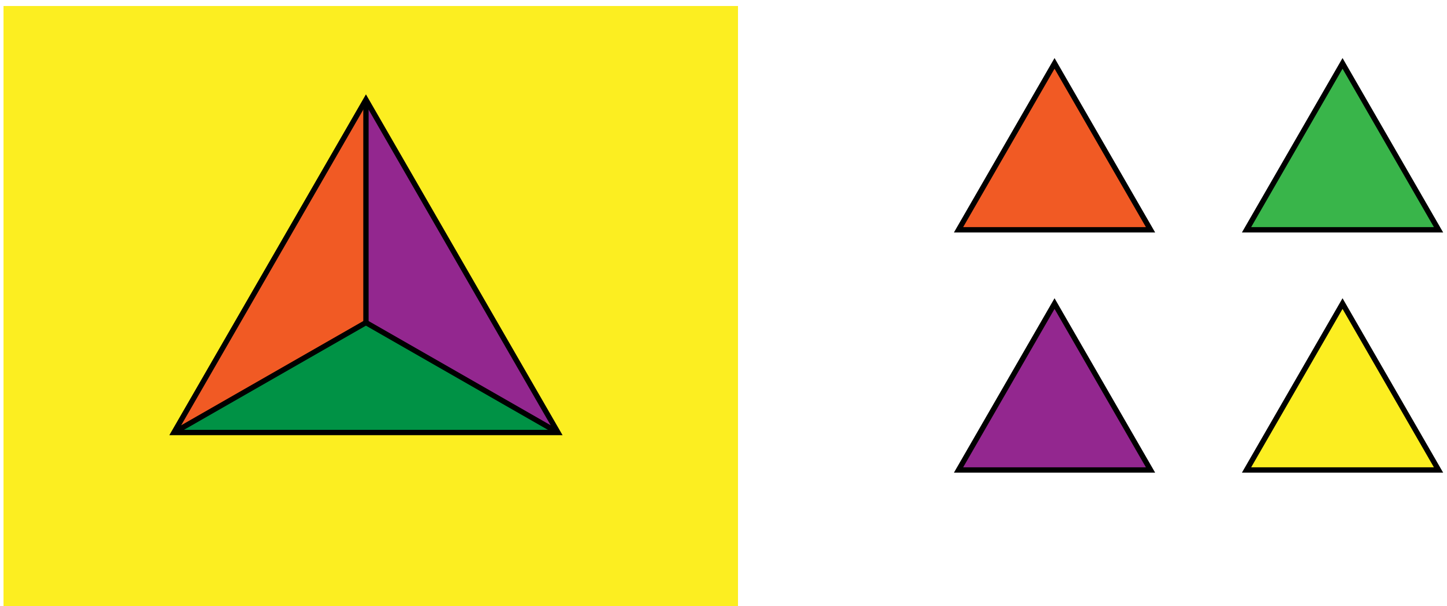

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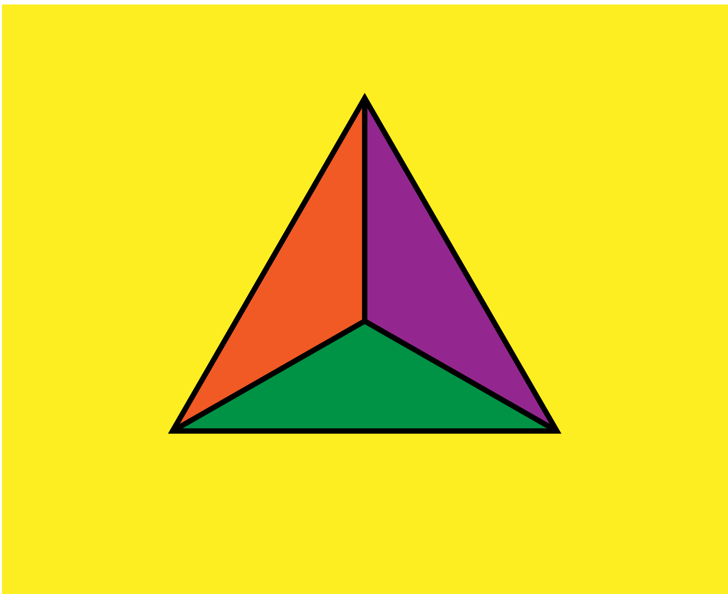
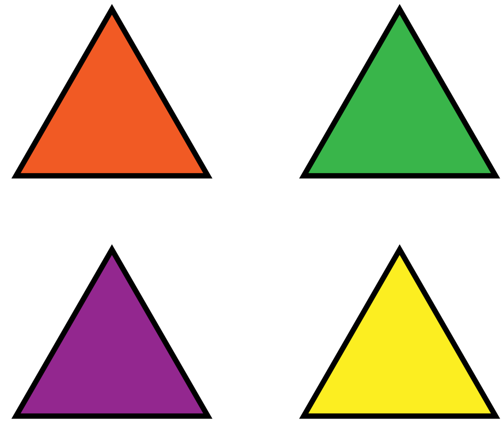
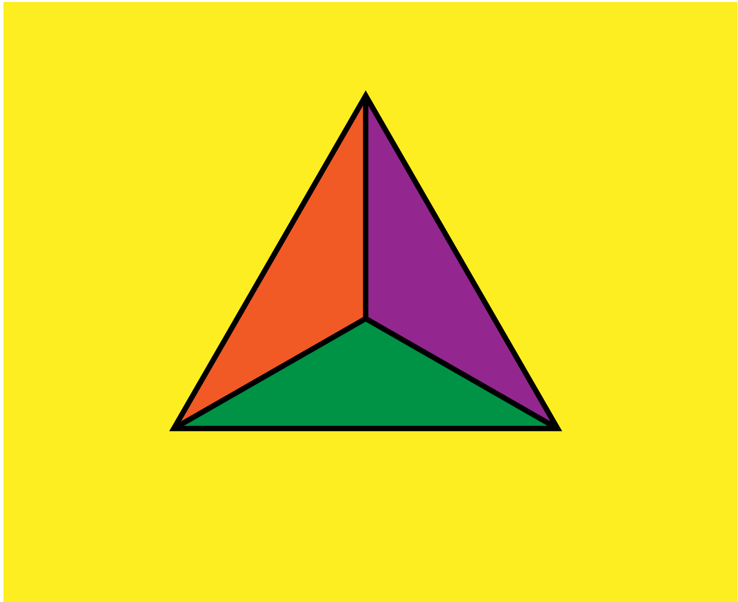
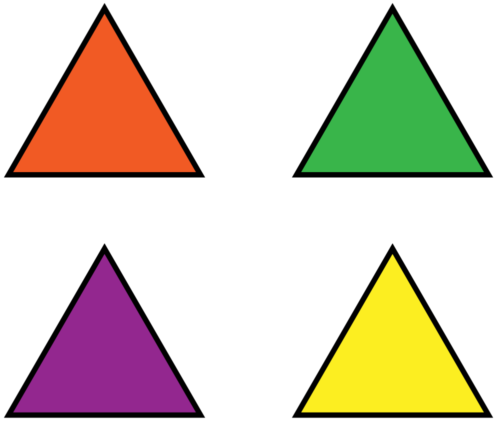




$F = 4$

Four Colors Needed

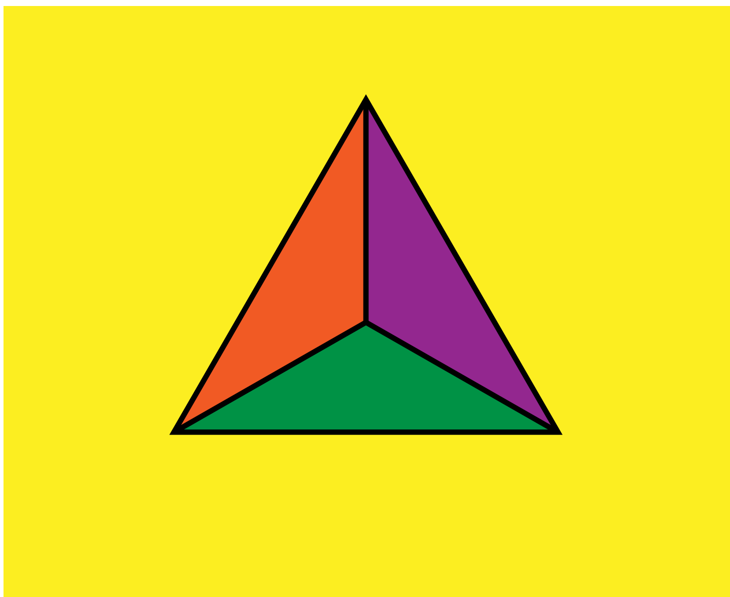


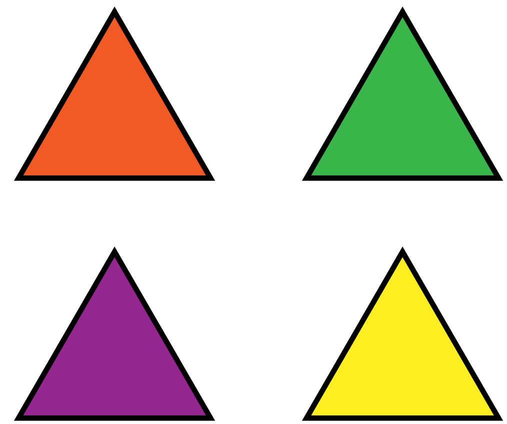


$F = 4$

$E = 12/2 = 6$

Four Colors Needed





$F = 4$

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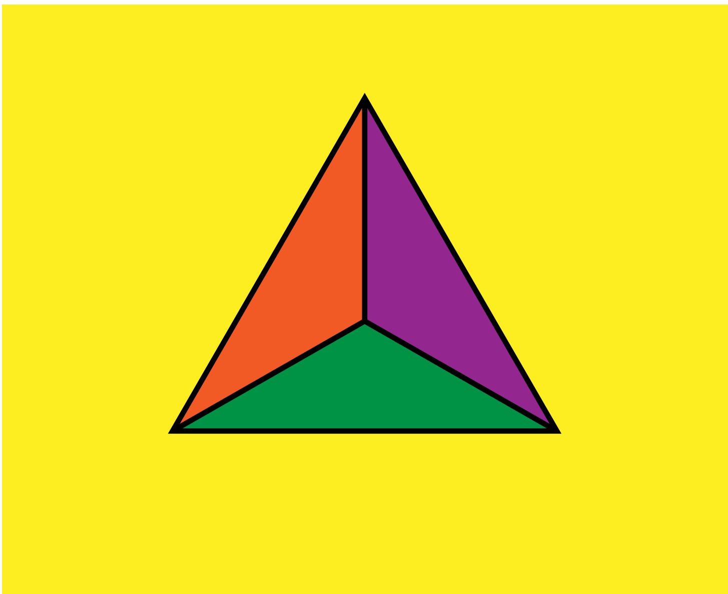
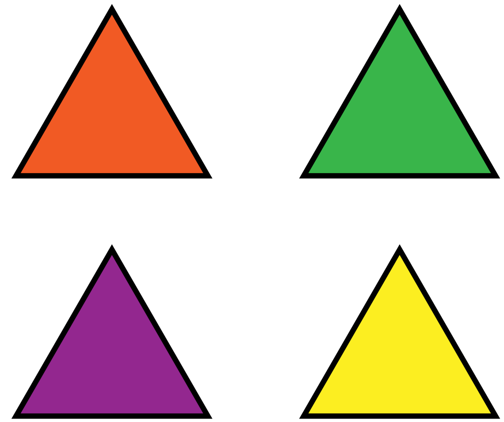
$V = 12/3 = 4$

Four Colors Needed





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Five Colors Needed?

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A	B	C	D	G

$F = 5$

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A	B	C	D	G

$F = 5$

$V \leq 6$

Five Colors Needed?

A	B	C	D	G

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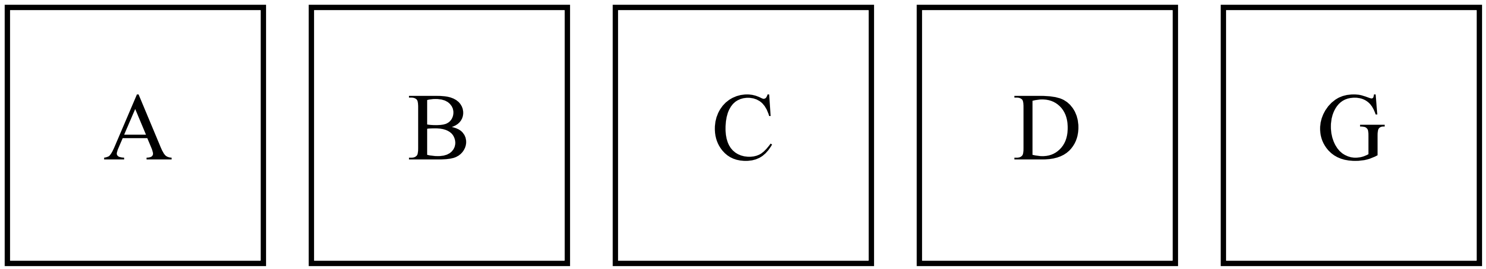
$V \leq 6$

$E \leq 9$

Five Colors Needed?



$F = 5$

$V \leq 6$

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A borders B,C,D,G 4 edges
B borders C,D,G 3 edges
C borders D,G 2 edges

Five Colors Needed?

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We already have 9 edges,
so D cannot border G!

Thus, there is no map with 5 regions,
each of which touches all the others.

Five Colors Needed?

A | B | C | D | G

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Thus, there is no map with 5 regions,
each of which touches all the others **on a sphere**.

Euler Characteristics

For the plane/sphere, $\chi = V - E + F = 2$

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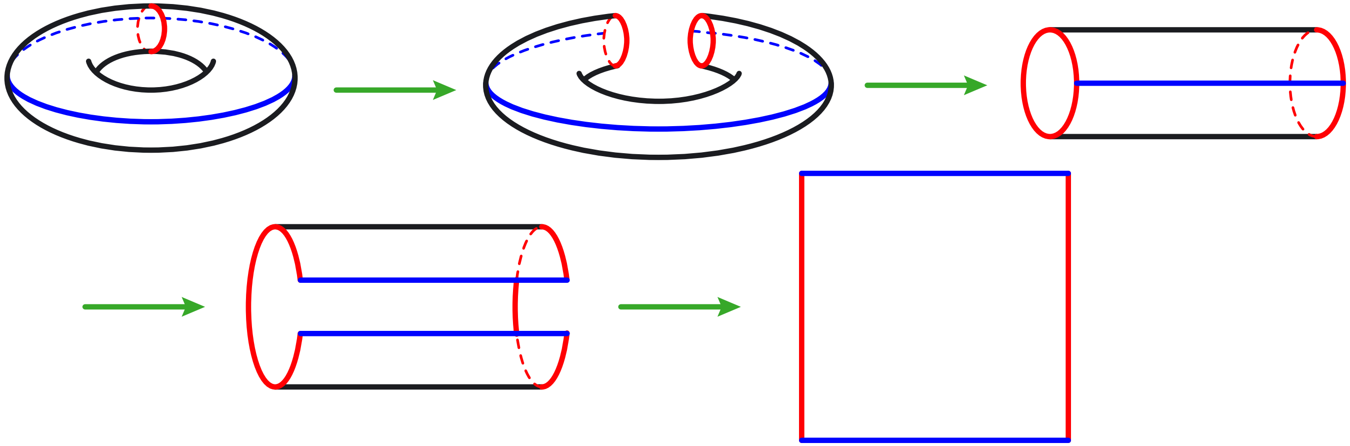

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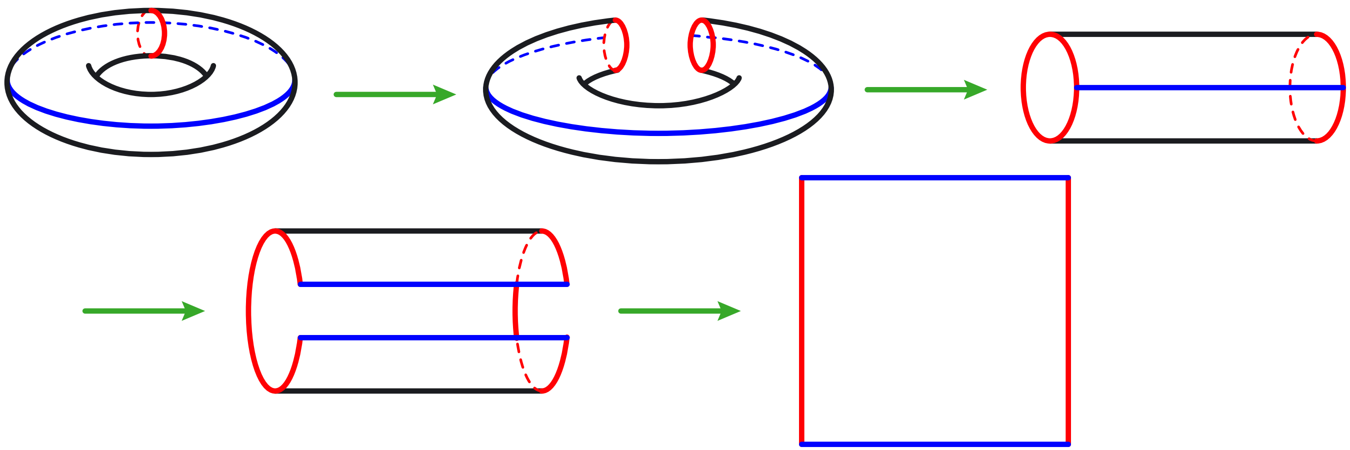

For the torus, $\chi = V - E + F = 1 - 2 + 1 = 0$

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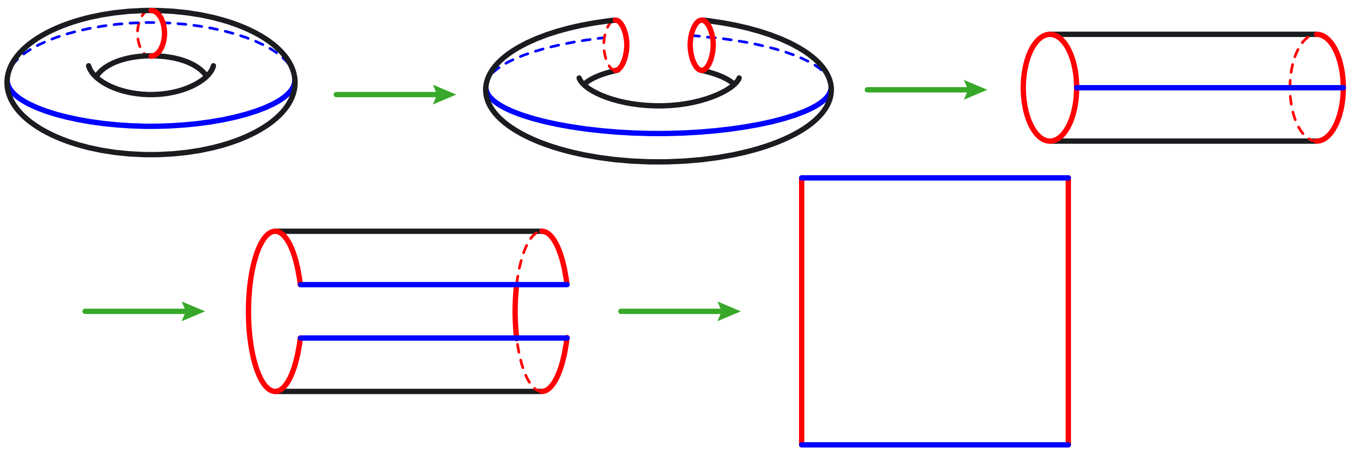

For the torus, $\chi = V - E + F = 1 - 2 + 1 = 0$

For the g-holed torus, $\chi = V - E + F = 2 - 2g$

Analyzing the Seven-Color Map

7 hexagons

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$(7 \times 6)/2 = 21$ edges

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$(7 \times 6)/2 = 21$ edges
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$(7 \times 6)/3 = 14$ vertices

$\chi = V - E + F = 14 - 21 + 7 = 0$

This is a torus!

Riffing on the Seven-Color Map

Norton Starr

Sophie Sommer

Skona Brittain

Moira Chas

sarah-marie belcastro
& Carolyn Yackel

Ellie Baker
& Kevin Lee

The Heawood Formula

For a g-holed torus, the number of colors required to color an arbitrary map so that neighboring regions are different colors is

$$\gamma(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor$$

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For a double torus, we need **8** colors.

Seeking an Eight-Color Map

8 heptagons

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$V = 18$
The number of vertices counting multiplicities is $8 \times 7 = 56$

Seeking an Eight-Color Map

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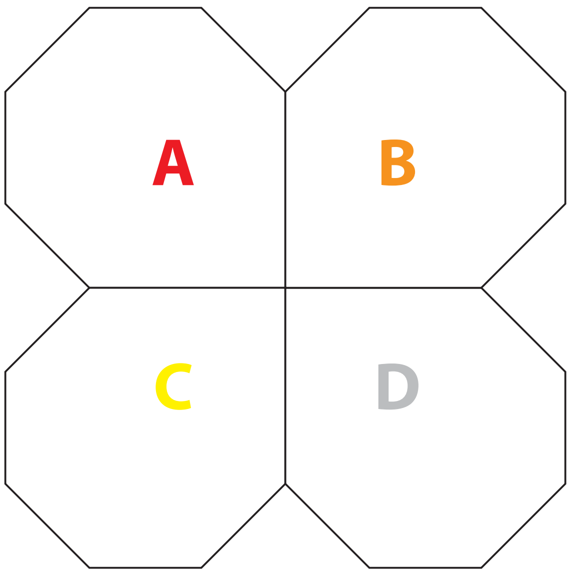
$(8 \times 7)/2 = 28$ edges

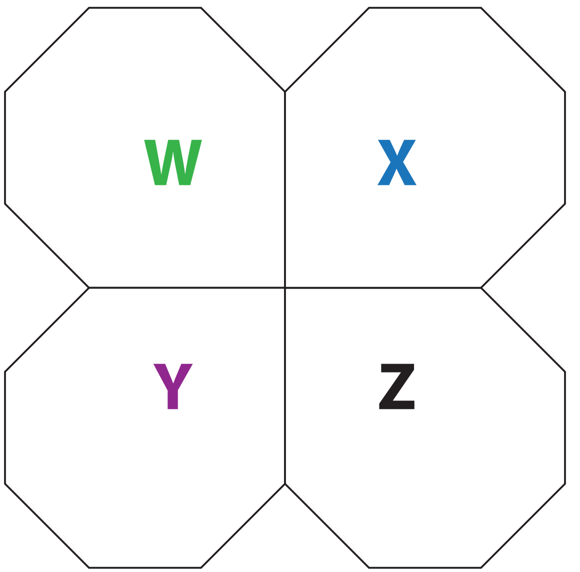
genus = 2
$\chi = -2 = V - E + F = V - 20$

$V = 18$
The number of vertices counting multiplicities is $8 \times 7 = 56$

Most symmetric resolution:
two vertices with 4 heptagons,
the rest with 3 heptagons

Fun with Heptagons





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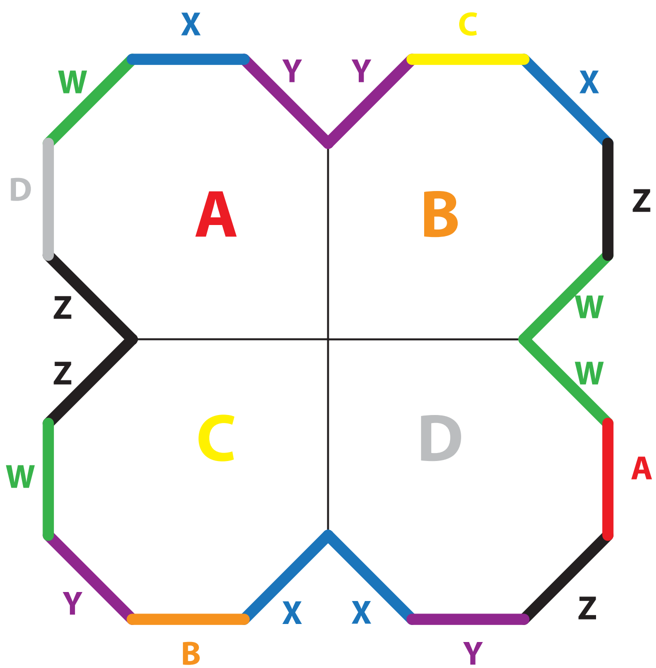
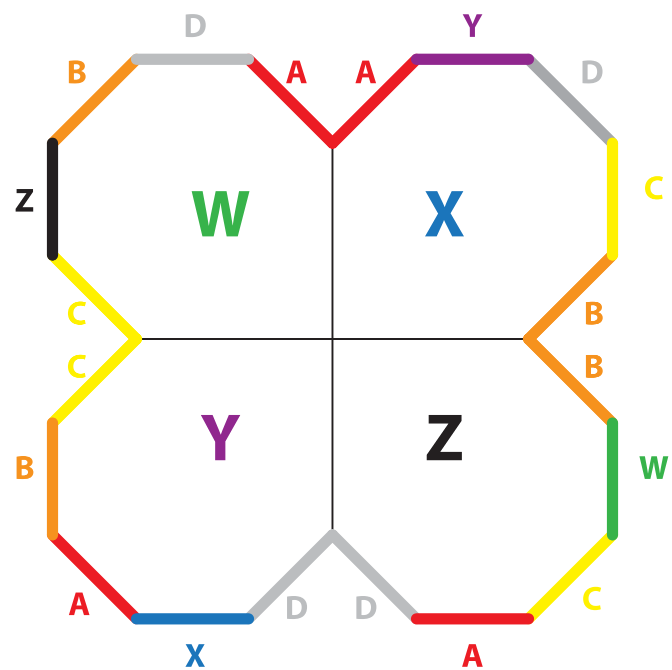
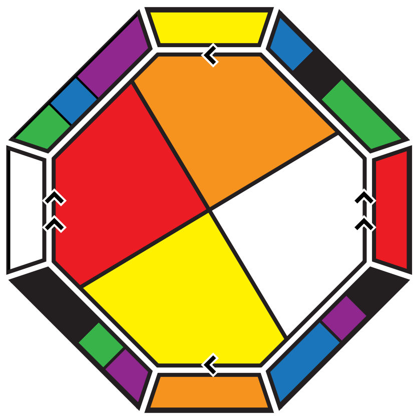
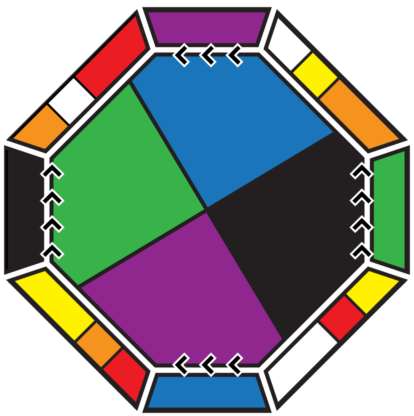




An Eight-Color Map Framework



punctured torus



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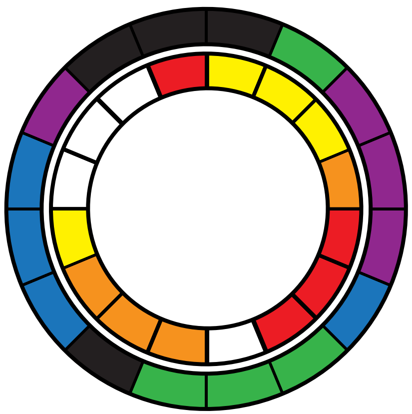

the colors at the boundary of the two punctures