Spin Representations and Lattices

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Abstract

We give a construction of the spin representations of orthogonal groups over $\mathbf{Z}$ that are compact over $\mathbf{R}$. Such an orthogonal group is associated to a positive definite, integral lattice L with good reduction at all primes. A representation of the group scheme $\mathrm{Spin}(L)$ is an even, unimodular lattice M of rank 2^k, where $k = n-1$ if $\mathrm{rank}(L) = 2n$ is even and $k = n$ if $\mathrm{rank}(L) = 2n+1$ is odd. To construct these representations, we use the properties of the Clifford algebra $C(L)$, and to obtain their even, unimodular structure, we use the properties of the Lie algebra $\mathfrak{spin}(L)$, which is a sublattice of finite index in $\mathfrak{so}(L) = \Lambda^2(L)$.

Let L be a positive definite, integral lattice with good reduction at all primes. In other words, L is positive definite, even, and has determinant 1 or 2. In this paper, we show that a spin representation of such a lattice is itself a positive definite, even, unimodular lattice. If $\mathrm{rank}(L) = 2n$ is even, then L has two half-spin representations of rank 2^{n-1}, and if $\mathrm{rank}(L) = 2n+1$ is odd, then L has one spin representation of rank 2^n.

In section 1, we examine the local structure of the lattice L. In particular, we find that $L_p = L \otimes \mathbf{Z}_p$ is a split lattice for all primes p. This in turn implies that the group $\mathrm{Spin}(L)$ is a model over $\mathbf{Z}$ of the group $\mathrm{Spin}(L \otimes \mathbf{Q})$ in the sense of [9]. Furthermore, if L has even rank, then the Clifford algebra $C(L_p)$ is split, and if L has odd rank, then the even part of the Clifford algebra $C_0(L_p)$ is split. Section 2 outlines the construction of the group $\mathrm{Spin}(L)$ inside $C_0(L)$,

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which is analogous to the classical vector-space construction, and describes the resulting representations of $\mathrm{Spin}(L_p)$ inside $C_0(L_p)$.

Our scenario is similar to that of a globally irreducible representation, as described by Gross in [8]. For a rational representation $\rho\colon G \to \mathrm{End}(W)$ that is absolutely irreducible, the criterion that ρ be globally irreducible is equivalent to the following scenario, originally described by Thompson in [13]:

> If M is a full rank $\mathbf{Z}$-submodule of W that is stable under the action of G, then the reduced representation sending G to $\mathrm{End}(M/pM)$ is irreducible for every prime p.

In this case, Thompson observed that M carries a uniquely determined structure of a positive definite, even, unimodular lattice whose inner product is invariant under the action of G. However, Thompson also observed that this phenomenon is quite rare. In [8], Gross demonstrates that there are many more examples of interesting lattices that arise from globally irreducible representations that are not absolutely irreducible. Further applications and amplifications of the globally irreducible lattice construction can be found in the papers [14–16] of Tiep, the first of which extends Gow's results in [7] concerning the lattices arising from basic spin representations of the double covers of the finite groups S_n and A_n.

In our case, the analogue of global irreducibility is provided by the Lie group structure of $\mathrm{Spin}(L)$. In section 3, we compute the Lie algebra $\mathfrak{spin}(L)$ of $\mathrm{Spin}(L)$, which is an index 2 sublattice of $\mathfrak{so}(L)$. Since $\mathfrak{spin}(L_p) = \mathfrak{spin}(L) \otimes \mathbf{Z}_p$, the spin representations of $\mathfrak{spin}(L_p)$, together with the spin representation of $\mathfrak{spin}(L \otimes \mathbf{R})$, yield corresponding spin representations of $\mathfrak{spin}(L)$ and $\mathrm{Spin}(L)$. In section 4, we construct spin representations over $\mathbf{Z}$ and demonstrate that they are endowed with an even, unimodular, $\mathrm{Spin}(L)$-invariant inner product.

As in the globally irreducible construction, we are left with the open question of how to determine *which* even unimodular lattice M results from our construction. To be more precise, we do not know of a general algorithm for determining the full automorphism group of the lattice M, or for computing invariants such as its minimal norm or theta series. However, in the event that L has a large automorphism group, it is possible to identify the constructed lattice in specific cases. For example, when $L = D_{8k}^{+}$, the resulting half-spin lattice is the Barnes-Wall lattice BW_{4k-1}, an identity which we will demonstrate in a forthcoming paper.

1 Locally split lattices

The following conventions hold throughout this paper.

R is a principal ideal domain.

K is its field of fractions.

L is a lattice over R, i.e., a finite-rank free R-module

carrying a quadratic form $q\colon R \to K$.

V is the finite-dimensional quadratic space $L \otimes_R K$.

In practice, R will always be $\mathbf{Z}$, $\mathbf{Z}_p$, or a field. We will usually require the quadratic form on L to take values in R; in particular, unless stated otherwise, we require lattices over $\mathbf{Z}$ to be even.

We denote by H the hyperbolic plane over R, i.e., the lattice with basis $\{e_1, e_{-1}\}$ such that $q(e_1) = q(e_{-1}) = 0$ and $\langle e_1, e_{-1}\rangle = 1$. The lattice L is **split** over R if L is a non-degenerate R-lattice that contains H^n as an orthogonal sublattice, where n is the greatest integer less than or equal to $\mathrm{rank}(L)/2$. When L is split, we choose bases $\{e_1, e_{-1}\}, \ldots, \{e_n, e_{-n}\}$ for each of the n hyperbolic planes in H^n, and in the odd rank case we let e_0 generate the orthogonal complement of H^n. The result is a basis $\{e_i\}$ for L such that $\langle e_i, e_j\rangle = \delta_{i,-j}$ unless $i = j = 0$. Under this convention, we let N be the maximal isotropic subspace spanned by $\{e_1, \ldots, e_n\}$ and N' be the maximal isotropic subspace spanned by $\{e_{-n}, \ldots, e_{-1}\}$.

Our focus is the case in which L is a positive definite lattice over $\mathbf{Z}$ with good reduction at all primes. The following proposition gives the local structure of such a lattice.

Proposition 1 *Let L_p be an integral lattice over $\mathbf{Z}_p$ that is even if $p = 2$.*

(1) If $\mathrm{rank}(L_p) = 2n$ and $\det(L_p) \equiv (-1)^n$, then L_p is split.
If $\mathrm{rank}(L_p) = 2n$, $\det(L_p) \not\equiv (-1)^n$, and $\det(L_p) \in \mathbf{Z}_p^*$, then $L_p = H^{n-1} \perp M$, where M is an anisotropic, unimodular lattice of rank 2.

(2) If $\mathrm{rank}(L_p) = 2n+1$ and $\det(L_p) \equiv 2a$ for some $a \in \mathbf{Z}_p^*$, then L_p is split.

This proposition follows from the classification of local lattices in [11, Sections 92–93]. When p is odd, it boils down to the fact that the isomorphism class of a $\mathbf{Z}_p$-lattice L of given rank with unit determinant is given by whether or not its determinant is a square in in $\mathbf{Z}_p$. When $p = 2$, a small calculation is required for rank 3 lattices, and the rest follows.

If L is a $\mathbf{Z}$-lattice with good reduction at all primes, then L must be even and have determinant ± 1 or ± 2. The added condition that L be positive definite (cf [12]) yields the following result.

If L is an integral, positive definite lattice over $\mathbf{Z}$,

$$L \text{ is even and } \begin{cases} \text{rank}(L) \equiv 0 \pmod 8 \text{ and } \det(L) = 1 \\ \text{or } \text{rank}(L) \equiv \pm 1 \pmod 8 \text{ and } \det(L) = 2 \end{cases}$$

$$\Updownarrow$$

$$L \text{ has good reduction for all } p$$

$$\Updownarrow$$

$$L \otimes \mathbf{Z}_p \text{ is split with good reduction for all } p.$$

We may define the Clifford algebra of a lattice L in the usual manner by setting $C(L) = T(L)/I(L)$, where $T(L)$ is the tensor algebra of L and $I(L)$ is the ideal generated by all elements of the form $x \otimes x - q(x)$. We denote the $\mathbf{Z}/2\mathbf{Z}$-graded pieces of the Clifford algebra of L by $C_0(L)$ and $C_1(L)$. It is a straightforward but tedious exercise in abstract nonsense to demonstrate that if S is any R-algebra, then $C(L \otimes_R S)$ is canonically isomorphic to $C(L) \otimes_R S$. As a result, the Clifford algebra construction passes naturally through localizations and reductions of the coefficient ring of L. It is also a simple exercise to show that if M is a sublattice of L, then $C(M)$ is the subalgebra of $C(L)$ generated by M.

In case L is split, we take maximal isotropic sublattices N and N' of L as above and observe that $C(N) = \Lambda(N)$ is a subalgebra of the Clifford algebra of L. When L has even rank, we may identify $C(N)$ with the left ideal $C(L)(e_{-1} \cdots e_{-n})$ and allow $C(L)$ to act on this ideal by right multiplication. A similar action of $C_0(L)$ on $C(N)$ may be constructed in the odd-rank case. An argument for the following theorem based on these constructions, as well as a thorough discussion of the properties of Clifford algebras (albeit not over rings), can be found in [3].

Theorem 2 *Suppose that L is a split lattice, and that N is a maximal isotropic sublattice. Let $M = C(N) = \Lambda(N)$; the $\mathbf{Z}/2\mathbf{Z}$-graded pieces of M will be denoted M_0 and M_1.*

(1) If $L = H^n$, then $C(L) = \text{End}(M)$ and

$$C_0(L) = \text{End}(M_0) \oplus \text{End}(M_1).$$

The module M_0 is the **even half-spin representation** *of $C_0(L)$ and the module M_1 is the* **odd half-spin representation** *of $C_0(L)$.*

(2) If $L = H^n \perp Re_0$*, where* $q(e_0) \in R^*$*, then* $C(L) = C_0(L) \oplus e_0 C_0(L)$ *and*

$$C_0(L) = \mathrm{End}(M).$$

The module M *is the* **spin representation** *of* $C_0(L)$*.*

Since $V \otimes \bar{K}$ is always split when V is non-degenerate, we have the following standard corollary.

Corollary 3 *Suppose that* V *is a non-degenerate quadratic space.*

(1) If V *is even-dimensional, then* $C(V)$ *is a central simple algebra. Furthermore, the center of* $C_0(V)$ *has dimension* 2 *over* K*, and* $C_0(V)$ *is either simple or the direct sum of two simple algebras.*
(2) If V *is odd-dimensional, then* $C_0(V)$ *is a central simple algebra. Furthermore, the center of* $C(V)$ *has dimension* 2 *over* K*, and* $C(V)$ *is either simple or the direct sum of two simple algebras.*

2 Orthogonal Groups and Spin Groups

The orthogonal group $\mathrm{O}(L)$ is the group of linear transformations preserving the quadratic form on L. When R does not have characteristic 2, the special orthogonal group $\mathrm{SO}(L)$ is the subgroup of $\mathrm{O}(L)$ consisting of matrices with determinant 1. In characteristic 2, $\mathrm{SO}(L)$ is defined using the Dickson determinant (cf [5]), which maps $\mathrm{O}(L)$ to $\mathbf{Z}/2\mathbf{Z}$ and whose kernel is $\mathrm{SO}(L)$.

One easily computes that the algebraic group $\mathrm{SO}(H)$ is the multiplicative group $\mathbf{G}_m$. As a consequence (cf [1, Section V.23]), the group $\mathrm{SO}(V)$ is split if and only if the quadratic space V is split.

Since $\mathrm{SO}(L)$ is the kernel of a homomorphism from $\mathrm{O}(L)$ to a group of order 2, the index of $\mathrm{SO}(L)$ in $\mathrm{O}(L)$ is at most 2. If v is an element of L such that $q(v)$ is a unit, we denote by r_v the reflection along v

$$r_v(x) = x - \frac{\langle x, v\rangle}{q(v)} v. \tag{1}$$

The theorem of Cartan and Dieudonné states that unless V is a split quadratic space of dimension 4 over the field of two elements, the group $O(V)$ is generated by reflections. In particular, $\mathrm{SO}(V)$ has index 2 in $O(V)$. On the other hand, in an arbitrary lattice, there may be no element v with the property that $q(v)$ is a unit, and so $\mathrm{O}(L)$ may contain no reflections and may be equal to $\mathrm{SO}(L)$. The Leech lattice $\Lambda_{24} \subseteq \mathbf{Q}^{24}$ is the least-rank unimodular example of this phenomenon.

To construct the group $\mathrm{Spin}(L)$, we use the algebra $C(L)$. As for the orthogonal group, $\mathrm{Spin}(L)$ is somewhat less tractable than $\mathrm{Spin}(V)$. We outline the lattice construction below.

Definition 4 *The* **main antiautomorphism of** $C(L)$ *is the antiautomorphism* $\alpha\colon C(L) \to C(L)$ *such that if* $v_1, \ldots, v_k \in L$*,*

$$\alpha(v_1 \cdots v_k) = v_k \cdots v_1.$$

The following lemma is a simple exercise.

Lemma 5 *Suppose* $v_1, \ldots, v_k \in L$ *and let* $u = v_1 \cdots v_k$*. Then* u *is invertible if and only if* $\alpha(u)u$ *is a unit.*

Definition 6 *We denote by* $C^*(L)$ *the multiplicative subgroup of* $C(L)$ *given by*

$$C^*(L) = \{u \in C(L) \mid u \text{ is invertible and } uLu^{-1} \subseteq L\},$$

and we set $\mathrm{CSpin}(L) = C^*(L) \cap C_0(L)$*, the subgroup of even elements of* $C^*(L)$*. For any* $u \in C^*(L)$*, we let* $\theta_u\colon L \to L$ *be the linear transformation*

$$\theta_u(x) = uxu^{-1}$$

and we let $\Theta\colon C^*(L) \to \mathrm{End}(L)$ *be the homomorphism given by* $\Theta(u) = \theta_u$*.*

Given $u \in C^*(L)$, for all $x \in L$,

$$q(\theta_u(x)) = q(uxu^{-1}) = uxu^{-1}uxu^{-1} = ux^2u^{-1} = q(x),$$

and therefore Θ maps $C^*(L)$ into the orthogonal group of L. If $v \in L$ is an invertible element of $C(L)$, then $\theta_v = -r_v$. In the case of a quadratic space, this observation together with the theorem of Cartan and Dieudonné allows us to compute the image of Θ. A proof of the following proposition may be found in several of the results of [3, Section II.3].

Proposition 7 *Suppose that* V *is not a split quadratic space of dimension* 4 *over* $\mathbf{Z}/2\mathbf{Z}$*. Then* $\Theta(C^*(V)) = \mathrm{O}(V)$ *when* $\dim(V)$ *is even or* $\mathrm{char}(K) = 2$ *and* $\Theta(C^*(V)) = \mathrm{SO}(V)$ *when* $\dim(V)$ *is odd and* $\mathrm{char}(K) \neq 2$*. Furthermore, the kernel of the restriction of* Θ *to* $\mathrm{CSpin}(V)$ *is* K^**,* $\Theta(\mathrm{CSpin}(V)) = \mathrm{SO}(V)$*, and every element of* $\mathrm{CSpin}(V)$ *is of the form* $v_1 \cdots v_k$ *for some* $v_1, \ldots, v_k \in V$*.*

Definition 8 *The* **spin group of** L *is the group*

$$\mathrm{Spin}(L) = \{u \in \mathrm{CSpin}(L) \mid \alpha(u)u = 1\}.$$

As the condition $\alpha(u)u = 1$ implies that $\alpha(u) = u^{-1}$, and as $V \cap C(L) = L$, we find that

$$\mathrm{Spin}(L) = \mathrm{Spin}(V) \cap C_0(L). \tag{2}$$

The kernel of the projection from $\mathrm{Spin}(V)$ to $\mathrm{SO}(V)$ is the group μ_2 of square roots of 1. Furthermore, it can be shown that if $\dim(V) \geq 3$, then $\mathrm{Spin}(V)$ is a connected, simply connected linear algebraic group. However, in passing from $\mathrm{CSpin}(V)$ to the spin group, we lose the property that the K-points of the group necessarily cover the K-points of $\mathrm{SO}(V)$ because the normalization requires taking square roots.

Consider the hyperbolic plane with its usual basis $\{e_{-1}, e_1\}$. Since $e_{-1}e_1 + e_1e_{-1} = 1$, we find that $\{e_{-1}e_1, e_1e_{-1}\}$ is a basis for $C_0(H)$. Furthermore,

$$(ae_{-1}e_1 + be_1e_{-1})(ce_{-1}e_1 + de_1e_{-1}) = ace_{-1}e_1 + bde_1e_{-1}.$$

As a consequence, we find that

$$\mathrm{CSpin}(H) = \{ae_{-1}e_1 + be_1e_{-1} \mid a, b \in K^*\}, \tag{3}$$

and that the action of $\mathrm{CSpin}(H)$ on H is given by

$$\Theta(ae_{-1}e_1 + be_1e_{-1}) = \begin{pmatrix} ab^{-1} & 0 \\ 0 & a^{-1}b \end{pmatrix}. \tag{4}$$

Since $\alpha(ae_{-1}e_1 + be_1e_{-1})(ae_{-1}e_1 + be_1e_{-1}) = ab$, we conclude that

$$\mathrm{Spin}(H) = \{ae_{-1}e_1 + a^{-1}e_1e_{-1} \mid a \in R^*\}. \tag{5}$$

Therefore $\mathrm{Spin}(H) \cong R^*$, and given that $\mathrm{SO}(H)$ is also R^*, we find that the image of $\mathrm{Spin}(H)$ in the special orthogonal group is $(R^*)^2$. As a consequence of this calculation, we may see directly that $\mathrm{Spin}(V)$ is split if and only if V is split.

In Theorem 2, we constructed the spin representations of $C_0(L)$ for split L. Restricting these representations to $\mathrm{Spin}(L)$ yields spin representations of the spin group of L in the split case. In the final section of this paper, we will construct spin and half-spin representations of anisotropic lattices.

In [3, Section 2.4], it is proven that if V is a non-degenerate quadratic space and v_0 is an anisotropic vector of V, then the set

$$\{v \in V \mid q(v) = q(v_0)\}$$

spans V unless $\dim(V) = 2$, V is split, and K has either 2 or 3 elements. From this result, we deduce the following lemma.

Lemma 9 *Unless V is the hyperbolic plane over the field of 2 elements,* $\mathrm{Spin}(V)$ *generates* $C_0(V)$ *as an algebra.*

Proof. Suppose that V is neither of the exceptional cases noted in the preceding paragraph, and choose an anisotropic $v_0 \in V$. If $a_0 = q(v_0)$, then the set

$$S = \{v \in V \mid q(v) = a_0\}$$

spans V, and thus

$$T = \{v_1 v_2 \mid q(v_1) = q(v_2) = a_0\}$$

is a set of generators for $C_0(L)$. The lemma follows from the fact that if $v_1 v_2 \in T$, then $v_1 v_2 / a_0 \in \mathrm{Spin}(V)$. The case in which V is the hyperbolic plane over $\mathbf{Z}/3\mathbf{Z}$ may be confirmed by explicit computation. $\square$

Corollary 10 *Unless V is the hyperbolic plane over the field of* 2 *elements, if* $\dim(V)$ *is even then the half-spin representations of* $\mathrm{Spin}(V)$ *are irreducible, and if* $\dim(V)$ *is odd then the spin representation of* $\mathrm{Spin}(V)$ *is irreducible.*

By contrast, when L is an anisotropic $\mathbf{Z}$-lattice, it is often the case that the spin representations of $\mathrm{Spin}(L)$ of the type we will construct in Section 6 are not irreducible. This possibility is evident when we consider the fact that there exist even unimodular lattices whose special orthogonal groups are $\{\pm 1\}$. However, if M is a spin or half-spin representation module for L and we consider the action of $\mathrm{Spin}(L/pL)$ on M/pM, the preceding corollary implies that this representation is irreducible for all primes p. To exploit this phenomenon, we now study the spin representations of the Lie algebra of $\mathrm{Spin}(L)$, which has a simpler relationship to its reduction modulo p than the Lie group does.

3 Orthogonal Lie Algebras

To define the Lie algebra of an algebraic group over an arbitrary ring R, one interprets it as the tangent space at the identity, which can be computed purely algebraically. For a complete description of the construction and its properties, see [4].

Definition 11 *Let G be a group over R, let $R[\epsilon]$ be the R-algebra defined by the relation $\epsilon^2 = 0$, and let $\pi\colon G(R[\epsilon]) \to G(R)$ be the map induced by the projection from $R[\epsilon]$ to R that sends $a + \epsilon b$ to a The* **Lie algebra of** *$G(R)$, denoted $\mathfrak{Lie}(G)(R)$, is $\pi^{-1}(e)$, where e is the identity of $G(R)$.*

Addition is inherited from the group operation on $G(R[\epsilon])$, scalar multiplication is induced by the homomorphism $s\colon R \to \mathrm{End}(R[\epsilon])$ given by

$$s(c)(a + \epsilon b) = a + \epsilon cb, \tag{6}$$

and the adjoint action of $G(R)$ arises from the inclusion $i\colon G(R) \to G(R[\epsilon])$ induced by the inclusion of R in $R[\epsilon]$, which allows us to set

$$\mathrm{Ad}(g)(x) = i(g)xi(g)^{-1}. \tag{7}$$

The Lie bracket on $\mathfrak{Lie}(G)(R)$ is defined as follows. As suggested by the notation for $\mathfrak{Lie}(G)(R)$, $\mathfrak{Lie}(G)$ acts as a functor from R-algebras to R-modules, and $\mathfrak{Lie}$ acts as a functor from groups over R to R-modules. If S is an R-algebra, then

$$\mathfrak{Lie}(G)(S) = \mathfrak{Lie}(G)(R) \otimes_R S. \tag{8}$$

The functor $\mathfrak{Lie}$ is left exact. In particular, given an inclusion $H \subseteq G$, we get a corresponding inclusion $\mathfrak{Lie}(H) \subseteq \mathfrak{Lie}(G)$. Applying the Lie functor to a morphism between groups over R allows us to differentiate homomorphisms of algebraic groups. Specifically, if $f\colon G \to H$ is a homomorphism, then we have the following commutative diagram, the rows of which are exact:

$$\begin{tikzcd} 1 \arrow[r] & \mathfrak{Lie}(G)(R) \arrow[r] \arrow[d, "df(R)"'] & G(R[\epsilon]) \arrow[r] \arrow[d, "f(R[\epsilon])"'] & G(R) \arrow[r] \arrow[d, "f(R)"'] & 1 \\ 1 \arrow[r] & \mathfrak{Lie}(H)(R) \arrow[r] & H(R[\epsilon]) \arrow[r] & H(R) \arrow[r] & 1 \end{tikzcd} \tag{9}$$

In this way, we may differentiate the adjoint map

$$\mathrm{Ad}\colon G(R) \to \mathrm{GL}(\mathfrak{Lie}(G)(R))$$

to get the adjoint map on the Lie algebra

$$\mathrm{ad}\colon \mathfrak{Lie}(G)(R) \to \mathrm{End}(\mathfrak{Lie}(G)(R)).$$

Hence, we may define the Lie bracket by the usual formula

$$[A, B] = \mathrm{ad}(A)B. \tag{10}$$

In the classical study of Lie algebras in characteristic 0, it would be odd to make a distinction between the Lie algebra of the special orthogonal group and that of the spin group, as the two algebras naturally coincide. However, when we work over fields of non-zero characteristic and over integral domains, we need to be more precise. As a matter of fact, we will soon see that if L is an even lattice over $\mathbf{Z}$, then the Lie algebra of $\mathrm{Spin}(L)$ is a proper submodule of the Lie algebra of $\mathrm{SO}(L)$. With the necessary distinction in mind, we denote $\mathfrak{Lie}(\mathrm{SO}(L))$ by $\mathfrak{so}(L)$ and we denote $\mathfrak{Lie}(\mathrm{Spin}(L))$ by $\mathfrak{spin}(L)$.

In characteristic other than 2, the Lie algebra of $\mathrm{SO}(L)$ is easily computed to be

$$\mathfrak{so}(L) = \{\phi \in \mathrm{End}(L) \mid \langle \phi x, y\rangle + \langle x, \phi y\rangle = 0 \text{ for every } x, y \in L\}. \tag{11}$$

In [1, Section V.23.6], it is computed that in characteristic 2, $\mathfrak{so}(L)$ contains the matrices A such that BA is antisymmetric (or equivalently, symmetric) subject to the additional requirement that the diagonal entries of BA vanish. This is equivalent to adding the constraint that $\langle \phi e_i, e_i\rangle = 0$.

In the previous section, we established an exact sequence

$$1 \longrightarrow \mu_2(R) \longrightarrow \mathrm{Spin}(L) \longrightarrow \mathrm{SO}(L).$$

Since the Lie functor is left exact, we conclude that we have an exact sequence

$$1 \longrightarrow \mathfrak{Lie}(\mu_2)(R) \longrightarrow \mathfrak{spin}(L) \longrightarrow \mathfrak{so}(L). \tag{12}$$

If $a \in R$ is in the Lie algebra of $\mu_2(R)$, then $(1 + \epsilon a)^2 = 1 + 2\epsilon a = 1$ and $2a = 0$. Therefore, in characteristic other than 2, the Lie algebra of $\mathrm{Spin}(L)$ maps injectively into the Lie algebra of $\mathrm{SO}(L)$, while in characteristic 2, the natural map from $\mathfrak{spin}(L)$ to $\mathfrak{so}(L)$ has a non-trivial kernel. To determine the image of $\mathfrak{spin}(L)$ in $\mathfrak{so}(L)$, more explicit computation is required. The Lie algebra of $\mathrm{SO}(V)$ is isomorphic to $\Lambda^2(V)$ (cf [6]); we will find that when L is a $\mathbf{Z}$-lattice of determinant 1 or 2, $\mathfrak{so}(L) = \Lambda^2(L)$ and $\mathfrak{spin}(L)$ is a submodule of index 2. We turn to the Clifford algebra to facilitate our calculations.

For the moment, we assume that the lattice L is unimodular; in particular, the following discussion will apply in the case of a quadratic space, and we will address the case in which L is not unimodular by using its inclusion in V. That L is unimodular implies that the bilinear form on L induces a canonical isomorphism of L and L^*. The endomorphism ring of any free R-module L is canonically isomorphic to $L \otimes L^*$, the action of $L \otimes L^*$ on L being generated by the formula

$$(x \otimes y^*)v = x(y^* v)$$

for any $x, v \in L$ and $y^* \in L^*$. Combining these isomorphisms, we obtain an identification of $\mathrm{End}(L)$ with $L \otimes L$. In order to express this identification concretely, we fix a basis $\{e_1, \ldots, e_l\}$ for L and let B be the corresponding matrix for the bilinear form on L. Since $\langle y, v\rangle = y^T B v$, the canonical map from L to L^* sends y to $y^T B$. Therefore, when $x, y \in L$,

$$x \otimes y \mapsto x y^T B \tag{13}$$

if we consider x and y as coordinate vectors. Using this formula, we easily verify that for any $A \in M_n(R)$,

$$A \mapsto \sum_{i,j} A e_i \otimes B^{-1} e_j. \tag{14}$$

We eliminate the matrices from this expression by noting that

$$\{B^{-1}e_1, \ldots, B^{-1}e_l\}$$

is the dual basis $\{e_1^*, \ldots, e_l^*\}$ of our chosen basis with respect to the bilinear form on L. Consequently, if $\phi \in \mathrm{End}(L)$, then

$$\phi \mapsto \sum_{i,j} \phi e_i \otimes e_j^*. \tag{15}$$

Equation 11 suggests that the image of $\mathfrak{so}(L)$ in $L \otimes L$ consists of the antisymmetric elements of the tensor product. More precisely, if $\bar{\alpha}$ is the endomorphism of $L \otimes L$ such that $\bar{\alpha}(x \otimes y) = y \otimes x$, then there is a natural isomorphism of the exterior product $\Lambda^2(L)$ with the submodule of elements w of the tensor product such that $\bar{\alpha}(w) = -w$. This identification is generated by the formula

$$x \wedge y = x \otimes y - y \otimes x;$$

that it is an isomorphism can be shown by choosing a basis for L. Note that in the general lattice case, while $\Lambda^2(L)$ thus imbedded is a direct summand of $L \otimes L$, it is not necessarily true that $L \otimes L = \Lambda^2(L) \oplus \mathrm{Sym}^2(L)$, since for instance, $x \otimes y = \frac{1}{2}[(x \otimes y + y \otimes x) + (x \otimes y - y \otimes x)]$.

Lemma 12 *If L is a unimodular lattice, then under the isomorphisms described above,*

$$\mathfrak{so}(L) = \Lambda^2(L).$$

The proof of this lemma is a routine calculation that follows from equations 13, 14, and 15.

Since $\Lambda^2(L) \cong \mathfrak{so}(L)$, the R-module $\Lambda^2(L)$ is endowed with the structure of a Lie algebra over R. With the identification above and some perserverance, we derive that

$$\begin{aligned}[w \wedge x, y \wedge z] = \langle w, z\rangle x \wedge y + \langle x, y\rangle w \wedge z \\ -\langle w, y\rangle x \wedge z - \langle x, z\rangle w \wedge y,\end{aligned} \tag{16}$$

and that for any $\phi \in \mathrm{SO}(L)$,

$$\mathrm{Ad}(\phi)x \wedge y = \phi x \wedge \phi y. \tag{17}$$

The Lie algebra $\Lambda^2(L)$ also carries a natural symmetric bilinear form defined by the formula

$$(w \wedge x, y \wedge z) = \det \begin{pmatrix} \langle w, y\rangle & \langle w, z\rangle \\ \langle x, y\rangle & \langle x, z\rangle \end{pmatrix}. \tag{18}$$

That this form is well-defined, bilinear and symmetric follows from the properties of the determinant. By Equation 17, we deduce that it is also invariant under the adjoint action of $\mathrm{SO}(L)$ and is thus a scaling of the Killing form on $\mathfrak{so}(L)$.

Our goal is to imbed the Lie algebra of the special orthogonal group into the Clifford algebra. To set the stage for this inclusion, we compute that for any $a, b, c, d \in L$,

$$\begin{aligned}[ab - ba, cd - dc] = 2\langle a, d\rangle(bc - cb) + 2\langle b, c\rangle(ad - da) \\ -2\langle a, c\rangle(bd - db) - 2\langle b, d\rangle(ac - ca).\end{aligned} \tag{19}$$

While this expression bears a promising resemblance to Equation 16, we now have an irksome factor of 2 to accommodate. Notably, this construction breaks down in characteristic 2, and we encounter difficulties when 2 is not invertible in R. For now, we assume that $2 \in R^*$ and map $L \otimes L$ into $C(L)$ by the formula

$$x \otimes y \mapsto \tfrac{1}{2}xy,$$

which yields a well-defined function of the tensor product since it is clearly linear in x and in y. This together with our previous isomorphisms gives the following sequence of maps:

$$\mathrm{End}(L) \longrightarrow L \otimes L^* \longrightarrow L \otimes L \longrightarrow C(L).$$

Denote the resulting map from $\mathrm{End}(L)$ to $C(L)$ by η.

Proposition 13 *Suppose that L is a unimodular R-lattice and that 2 is a unit in R. Let $\eta\colon \mathrm{End}(L) \to C(L)$ be defined as above, and let $C^2(L)$ be the submodule of $C(L)$ generated by products of two or fewer elements of L. Then the restriction of η to $\mathfrak{so}(L)$ is an injection, and the image of $\mathfrak{so}(L)$ in $C(L)$ consists of the even elements of $C^2(L)$ with trace 0, where the trace of $u \in C(L)$ is the trace of the endomorphism defined by left multiplication by u. Furthermore, η sends the Lie bracket on $\mathfrak{so}(L)$ to the Lie bracket on $C(L)$.*

Proof. To make explicit use of the identity $\mathfrak{so}(L) = \Lambda^2(L)$, we let $\eta'\colon \Lambda^2(L) \to C(L)$ be the endomorphism induced by η. For any $x, y \in L$,

$$\eta'(x \wedge y) = \tfrac{1}{2}(xy - yx) = xy - \tfrac{1}{2}\langle x, y\rangle. \tag{20}$$

If we let $\{e_1, \ldots, e_l\}$ be a basis for L, then $\{e_i \wedge e_j \mid i < j\}$ is a basis for $\Lambda^2(L)$. We find that

$$\{\eta'(e_i \wedge e_j) \mid i < j\} = \{e_i e_j + \tfrac{1}{2}\langle e_i, e_j\rangle \mid i < j\}$$

is a linearly independent subset of $C(L)$, and thence that η' is injective. It is evident from the formula above that the image of η' is contained in $C^2(L) \cap C_0(L)$, and moreover that

$$C^2(L) \cap C_0(L) = \eta'(\Lambda^2(L)) \oplus R. \tag{21}$$

As $\mathrm{trace}(1) = 2^l$, to demonstrate that $\eta'(\Lambda^2(L)) = C^2(L) \cap C_0(L) \cap \ker(\mathrm{trace})$ it suffices to show that any element of the image of η' has trace 0. In turn, it suffices to show that $\mathrm{trace}(e_i e_j - \frac{1}{2}\langle e_i, e_j\rangle) = 0$, a simple but tedious calculation that we omit.

The final assertion that η is a Lie algebra homomorphism follows immediately from a comparison of Equations 16 and 19. □

We may now relax our conditions on L and consider the consequences. We assume that $\mathrm{char}(R) \neq 2$. There are two possible obstructions to the previous imbedding of $\mathfrak{so}(L)$ in $C_0(L)$. Most obviously, if 2 is not invertible in R, then we cannot define the homomorphism from $\Lambda^2(L)$ to $C_0(L)$ as above. Moreover, if L is not unimodular, we may not have an isomorphism between $\mathfrak{so}(L)$ and $\Lambda^2(L)$ because L is a proper sublattice of L^*, and thus $L \otimes L^*$ does not map canonically to $L \otimes L$. In fact, the elements of $L \otimes L \subseteq L \otimes L^*$ are precisely those corresponding to endomorphisms that map L^* to L. However, we may use the method above to map $\mathfrak{so}(L)$ canonically into $\Lambda^2(V)$ in such a way that

$$\Lambda^2(L) \subseteq \mathfrak{so}(L) \subseteq \Lambda^2(L^*).$$

By a similar computation to that required for Lemma 12, we find that

$$\mathfrak{so}(L) \cap L \otimes L = \Lambda^2(L). \tag{22}$$

Similarly, we may now use the map $\eta' \colon \Lambda^2(V) \to C_0(V)$ to imbed $\mathfrak{so}(L)$ in $C_0(V)$ in a manner that preserves its Lie algebra structure, but in general $\mathfrak{so}(L) \cap C_0(L)$ will be a submodule of finite index in $\mathfrak{so}(L)$.

Now we may compare $\mathfrak{so}(L)$ to the Lie algebra of the spin group, which was constructed inside $C_0(L)$ in the first place. By the functorial properties of the Clifford algebra, $C(L \otimes R[\epsilon]) = C(L) \otimes R[\epsilon]$. Recall that

$$\mathrm{Spin}(L) = \{u \in C_0(L) \mid \alpha(u)u = 1, \text{ and } uLu^{-1} \subseteq L\}. \tag{23}$$

Suppose that $w \in C_0(L)$ is an element of $\mathfrak{spin}(L)$. The condition that

$$\alpha(1 + \epsilon w)(1 + \epsilon w) = 1$$

implies that

$$\alpha(w) + w = 0. \tag{24}$$

Moreover, since $(1 + \epsilon w)^{-1} = (1 - \epsilon w)$,

$$(1 + \epsilon w)v(1 - \epsilon w) = 1 + \epsilon(wv - vw) \in L \otimes R[\epsilon] = L + \epsilon L,$$

and thus

$$wL - Lw \subseteq L. \tag{25}$$

It is now a straightforward matter to show that if K does not have characteristic 2, then every element of $\eta(\mathfrak{so}(V))$ satisfies equations 24 and 25, and therefore $\mathfrak{so}(V) = \mathfrak{spin}(V)$. Note that since $\mathrm{rank}(\mathfrak{spin}(L)) = \dim(\mathfrak{spin}(V))$ by Equation 8, $\mathfrak{spin}(L)$ will always be a submodule of finite index in $\mathfrak{so}(L)$ provided that $\mathrm{char}(R) \neq 2$. Also, since $V \cap C(L) = L$ and since $\mathfrak{spin}(L) \subseteq \mathfrak{so}(L)$, we find that

$$\mathfrak{spin}(L) = \mathfrak{spin}(V) \cap C_0(L) = \mathfrak{so}(V) \cap C_0(L) = \mathfrak{so}(L) \cap C_0(L). \tag{26}$$

Armed with this equation, we may compare $\mathfrak{so}(L_p)$ with $\mathfrak{spin}(L_p)$ for split $\mathbf{Z}_p$-lattices.

Proposition 14 *If L_p is a split $\mathbf{Z}_p$-lattice with good reduction at p, then $\mathfrak{so}(L_p) = \Lambda^2(L_p)$. If $p \neq 2$, then $\mathfrak{spin}(L_p) = \mathfrak{so}(L_p)$. If $p = 2$, then $\mathfrak{so}(L_2)$*

is an odd lattice, and $\mathfrak{spin}(L_2)$ is the unique even sublattice of $\mathfrak{so}(L_2)$ of index 2.

Proof. Whenever $p \neq 2$ or $\mathrm{rank}(L_p)$ is even, L_p is unimodular, which implies that $\mathfrak{so}(L_p) = \Lambda^2(L_p)$.

Suppose that L_2 is a split $\mathbf{Z}_2$-lattice of odd rank. There is an $e \in L_2$ such that $L_2 = L_2' \perp \mathbf{Z}_2 e$ where L_2' is split and $q(e)$ is a unit. Note that $L_2^* = L_2' \perp \mathbf{Z}_2(\frac{1}{2}e)$.

Let $\phi \in \mathrm{End}(L_2)$ be an element of $\mathfrak{so}(L_2)$. We claim that under the canonical identification of $\mathrm{End}(L_2)$ and $L_2 \otimes L_2^*$, ϕ lies in $L_2 \otimes L_2$. To prove this, it suffices to show that $\phi(L_2^*) \subseteq L_2$, or equivalently that $\phi e \in 2L_2$. By Equation 11, for every $x \in L_2$,

$$\langle \phi e, x\rangle = -\langle e, \phi x\rangle \equiv 0 \pmod 2$$

since $e \in 2L_2^*$. Therefore, $\phi e \in 2L_2^*$. Since moreover $\langle \phi e, e\rangle = -\langle e, \phi e\rangle = 0$,

$$\phi e \in 2L_2^* \cap L_2' = 2L_2' \subset 2L_2.$$

Thus $\mathfrak{so}(L_2) \subseteq L_2 \otimes L_2$, and by Equation 22, $\mathfrak{so}(L_2) = \Lambda^2(L_2)$.

Let $V_p = L_p \otimes \mathbf{Q}_p$. If $p \neq 2$, then because $2 \in \mathbf{Z}_p^*$, $\Lambda^2(L_p) \subseteq C_0(L_p)$ and hence

$$\mathfrak{spin}(L_p) = \mathfrak{so}(V_p) \cap C_0(L_p) = \mathfrak{so}(L_p).$$

Now suppose that $p = 2$. Since the argument in both even and odd rank cases is essentially the same, we will present only the former. Since L_2 is split, we choose a basis $\{e_{-n}, \ldots, e_{-1}, e_1, \ldots, e_n\}$ for L_2 such that $\langle e_i, e_j\rangle = \delta_{i,-j}$. A basis for the image of $\Lambda^2(L_2)$ in $C_0(V_p)$ is

$$\{e_i e_j \mid i < j \text{ and } i \neq -j\} \cup \{e_{-i}e_i - \tfrac{1}{2} \mid 1 \leq i \leq n\}.$$

If $u \in \Lambda^2(L_2)$ and

$$u = \sum a_{i,j} e_i e_j + \sum a_i (e_{-i}e_i - \tfrac{1}{2}),$$

then the necessary and sufficient condition for u to be an element of $C_0(L_2)$ and thus of $\mathfrak{spin}(L_2)$ is that $\sum a_i \equiv 0 \pmod 2$. It follows that the index of $\mathfrak{spin}(L_2)$ in $\Lambda^2(L_2)$ is 2. A basis for the image of $\mathfrak{spin}(L_2)$ in $\Lambda^2(L_2)$ is

$$\{e_i \wedge e_j \mid i < j \text{ and } i \neq -j\} \cup \{2e_{-i} \wedge e_i \mid 1 \leq i < n\} \cup \{e_{-1} \wedge e_1 + e_{-n} \wedge e_n\}.$$

Since by equation 18,

$$\begin{aligned}
(e_i \wedge e_j, e_i \wedge e_j) &= 0,\\
(2e_{-i} \wedge e_i, 2e_{-i} \wedge e_i) &= -4,\\
(e_{-1} \wedge e_1 + e_{-n} \wedge e_n, e_{-1} \wedge e_1 + e_{-n} \wedge e_n) &= -2,
\end{aligned}$$

we conclude that $\mathfrak{spin}(L_2)$ is even. On the other hand, since

$$(e_{-1} \wedge e_1, e_{-1} \wedge e_1) = -1,$$

$\Lambda^2(L_2)$ is odd. $\square$

Corollary 15 *If L_p is a split $\mathbf{Z}_p$-lattice of rank at least 4 with good reduction modulo p, then $\mathfrak{spin}(L_p)$ generates $C_0(L_p)$ as an associative $\mathbf{Z}_p$-algebra.*

Proof. Let $C^2(L_p)$ be the submodule of $C(L_p)$ generated by products of two or fewer elements of L_p. Since $C^2(L_p) \cap C_0(L_p)$ generates $C_0(L_p)$ it suffices to show that $\mathfrak{spin}(L_p)$ generates $C^2(L_p) \cap C_0(L_p)$. (When $p \neq 2$, this follows directly from the previous two propositions.)

Choose a basis $\{e_{-n}, \dots, e_n\}$ for L_p as above, where we omit e_0 if $\operatorname{rank}(L_p)$ is even. By Proposition 14, $\mathfrak{spin}(L_p)$ contains e_ie_j for $i \neq \pm j$. Therefore, for each $i \in \{1, \dots, n\}$, we may take $j \in \{1, \dots, n\}$ such that $j \neq i$ and compute that

$$\begin{aligned} e_{-i}e_{-j} \cdot e_je_i + e_{-i}e_j \cdot e_{-j}e_i &= e_{-i}(e_{-j}e_j + e_je_{-j})e_i \\ &= e_{-i} \cdot \langle e_{-j}, e_j \rangle \cdot e_i \\ &= e_{-i}e_i. \end{aligned}$$

Since $\{1\} \cup \{e_ie_j \mid i < j\}$ is a basis for $C^2(L_p) \cap C_0(L_p)$, we conclude that $C^2(L_p) \cap C_0(L_p)$ is contained in the subalgebra generated by $\mathfrak{spin}(L_p)$. $\square$

Whenever we have a representation of the spin group of L, we may differentiate it to obtain a representation of the corresponding Lie algebra. Given a spin representation $\rho\colon \mathrm{Spin}(L) \to \mathrm{End}(M)$ defined by restricting a representation of $C_0(L)$, the spin representation $d\rho\colon \mathfrak{spin}(L) \to \mathrm{End}(M)$ is also defined by restricting the representation of $C_0(L)$. In a similar fashion, we may compute $d\rho^+$ and $d\rho^-$ for a lattice L of even rank by restricting the corresponding half-spin representations of $C_0(L)$. Corollary 15 implies that in the case of a split $\mathbf{Z}_p$-lattice L_p of rank at least 4 with good reduction, a spin representation of the Lie algebra $\mathfrak{spin}(L_p)$ that is the restriction of an irreducible representation of $C_0(L_p)$ is itself irreducible.

4 Spin representations of Z-lattices

In this section, L is an integral, even, positive definite lattice lying inside the vector space $\mathbf{Q}^l$ equipped with the standard inner product $\langle\cdot,\cdot\rangle$ and satisfying

$$\operatorname{rank}(L) \equiv 0 \pmod 8 \text{ and } \det(L) = 1$$
$$\text{or } \operatorname{rank}(L) \equiv \pm 1 \pmod 8 \text{ and } \det(L) = 2.$$

In addition, we adopt the following conventions.

q	is the quadratic form on L.
L_p	is the split $\mathbf{Z}_p$-lattice $L \otimes \mathbf{Z}_p$.
N_p	is a maximal isotropic subspace of L_p.
M_p	is $\Lambda(N_p)$, the Clifford algebra of N_p.
$M_{0,p}$	is the zeroth graded piece of M_p.
$M_{1,p}$	is the first graded piece of M_p.
V	is $L \otimes \mathbf{Q}$.
V_p	is $V \otimes \mathbf{Q}_p$.
V_∞	is $V \otimes \mathbf{R}$.
W_p	is $M_p \otimes \mathbf{Q}_p$.
$W_{0,p}$	is $M_{0,p} \otimes \mathbf{Q}_p$.
$W_{1,p}$	is $M_{1,p} \otimes \mathbf{Q}_p$.

If L has odd rank, we denote by ρ_p the spin representation $\rho_p\colon C_0(V_p) \to W_p$, and by abuse of notation we similarly denote its restrictions to $C_0(L_p)$ and to the spin groups and spin Lie algebras of V_p and L_p. If L has even rank, we likewise denote the various even half-spin representations by ρ_p^+ and the various odd half-spin representations by ρ_p^-. In this section, we will assemble these local data and apply them to the global case. We will find that even though L is manifestly non-split, the splitting of each local lattice L_p will imply that $C_0(L)$ is either a matrix algebra or a sum of matrix algebras, yielding spin representations associated to L.

To determine the Lie algebra of $\operatorname{Spin}(L)$, we consider the localization $\mathfrak{spin}(L_p)$.

Proposition 16 *The Lie algebra $\mathfrak{so}(L)$ is equal to $\Lambda^2(L)$, and the Lie algebra $\mathfrak{spin}(L)$ is the unique index 2 even sublattice of $\Lambda^2(L)$.*

Proof. The functorial properties of the Lie algebra construction (as in Equation 8) imply that $\mathfrak{so}(L_p) = \mathfrak{so}(L) \otimes \mathbf{Z}_p$ and $\mathfrak{spin}(L_p) = \mathfrak{spin}(L) \otimes \mathbf{Z}_p$, and as always $\mathfrak{so}(L)$ and $\mathfrak{spin}(L)$ are contained in $\mathfrak{so}(V) = \Lambda^2(V)$. By Proposition 14,

$$\mathfrak{so}(L) \otimes \mathbf{Z}_p = \Lambda^2(L_p) = \Lambda^2(L) \otimes \mathbf{Z}_p,$$

for all primes p, and therefore $\mathfrak{so}(L) = \Lambda^2(L)$. Similarly, since $\mathfrak{spin}(L_p)$ is contained in $\mathfrak{so}(L_p)$ with index 1 when $p \neq 2$ and with index 2 when $p = 2$, we conclude that $\mathfrak{spin}(L)$ is an index 2 sublattice of $\mathfrak{so}(L)$. The rest of the proposition follows from the fact that $\mathfrak{so}(L_2)$ is odd and $\mathfrak{spin}(L_2)$ is even, which implies the same of $\mathfrak{so}(L)$ and $\mathfrak{spin}(L)$. $\square$

A similar comparison of local data gives the following crucial result.

Proposition 17 *The Lie algebra $\mathfrak{spin}(L)$ generates $C_0(L)$ as an associative $\mathbf{Z}$-algebra.*

Proof. By Corollary 15, for each prime p the associative algebra generated by $\mathfrak{spin}(L_p) = \mathfrak{spin}(L) \otimes \mathbf{Z}_p$ is $C_0(L_p) = C_0(L) \otimes \mathbf{Z}_p$. Since $\mathfrak{spin}(L)$ generates $\mathfrak{spin}(L_p)$ as a $\mathbf{Z}_p$-module, the subalgebra of $C_0(L)$ generated by $\mathfrak{spin}(L)$ is $C_0(L)$ itself. $\square$

In order to construct representations of $\mathrm{Spin}(L)$ and $\mathfrak{spin}(L)$, we need to exploit the relationships between the lattices L and L_p and the quadratic spaces V, V_p and L/pL. In so doing, we use the fact that if L_1 and L_2 are lattices in $\mathbf{Q}^l$, for all but finitely many primes p, $L_1 \otimes \mathbf{Z}_p = L_2 \otimes \mathbf{Z}_p$.

The p-adic spin representation ρ_p of $C_0(L_p)$ may be reduced modulo p to give a representation

$$\bar{\rho}_p\colon C_0(L/pL) \to M_p/pM_p.$$

Of course, we can also construct $\bar{\rho}_p$ directly from Proposition 2 using the maximal isotropic subspace N_p/pN_p of L/pL, and we therefore observe that $\bar{\rho}_p$ is irreducible in the odd rank case. Similarly, in the even rank case, the half-spin representations may be reduced modulo p to give irreducible representations $\bar{\rho}_p^+$ and $\bar{\rho}_p^-$ mapping $C_0(L/pL)$ onto $\mathrm{End}(M_{0,p}/pM_{0,p})$ and $\mathrm{End}(M_{1,p}/pM_{1,p})$ respectively.

Proposition 18 *If L has odd rank and M_p' is a lattice in W_p that is stabilized by the action of $C_0(L_p)$, then $M_p' = p^k M_p$ for some integer k. If L has even rank, and M_p' is a lattice in W_p that is stabilized by the action of the full Clifford algebra $C(L_p)$, then $M_p' = p^k M_p$ for some integer k; if $i = 0$ or 1, and $M_{i,p}'$ is a lattice in $W_{i,p}$ that is stabilized by the action of $C_0(L_p)$, then $M_{i,p}' = p^k M_{i,p}$ for some integer k.*

Proof. Suppose that $\mathrm{rank}(L)$ is odd. Scaling by a power of p if necessary, we may assume that the stable sublattice M'_p is a subset of M_p but not of pM_p. Consequently, $M'_p/(pM_p \cap M'_p)$ is a non-trivial vector subspace of M_p/pM_p which is stabilized by the action of $C_0(L/pL)$. Since the reduced representation of $C_0(L/pL)$ is irreducible, we conclude that

$$M'_p/(pM_p \cap M'_p) = M_p/pM_p.$$

Therefore, Nakayama's Lemma implies that $M'_p = M_p$. A similar argument treats the even rank case. □

Let us suppose for the moment that $\operatorname{rank}(L) = \dim(V) = 2n+1$ is odd. Since V_p is split for all p, Theorem 2 implies that $C_0(V_p)$ is a matrix algebra over $\mathbf{Q}_p$. Class field theory now allows a similar conclusion for $C_0(V)$ using the exact sequence of Brauer groups (cf [2])

$$0 \longrightarrow \operatorname{Br}(\mathbf{Q}) \longrightarrow \bigoplus_p \operatorname{Br}(\mathbf{Q}_p) \oplus \operatorname{Br}(\mathbf{R}) \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow 0, \tag{27}$$

where $\operatorname{Br}(\mathbf{Q})$ maps into $\operatorname{Br}(K)$ by tensoring with K and $\bigoplus_p \operatorname{Br}(\mathbf{Q}_p) \oplus \operatorname{Br}(\mathbf{R})$ maps onto $\mathbf{Q}/\mathbf{Z}$ by taking the sum of the coordinates.

Standard results (cf. [10, Section I.4]) imply that since $2n \equiv 0, 6 \pmod 8$, $C_0(V_\infty) \cong C(\mathbf{R}^{2n})$ is a matrix algebra. Alternatively, we may deduce that $C_0(V_\infty)$ is a matrix algebra using the exact sequence in Equation 27. Since the equivalence class of $C_0(V) \otimes \mathbf{Q}_p = C_0(V_p)$ in $\operatorname{Br}(\mathbf{Q}_p)$ is the identity, the injectivity of $\operatorname{Br}(\mathbf{Q})$ implies that $C_0(V)$ belongs to the identity class in $\operatorname{Br}(\mathbf{Q})$. Therefore, there exists a $\mathbf{Q}$-vector space W of dimension 2^n which is a simple $C_0(V)$-module under an isomorphism $\rho\colon C_0(V) \to \operatorname{End}(W)$. Since there are canonical isomorphisms between $\operatorname{End}(W) \otimes \mathbf{Q}_p$ and $\operatorname{End}(W \otimes \mathbf{Q}_p) = \operatorname{End}(W_p)$ and between $C_0(V) \otimes \mathbf{Q}_p$ and $C_0(V_p)$, we conclude that $W \otimes \mathbf{Q}_p$ is isomorphic to W_p as a $C_0(V_p)$-module. Similarly, if we set $W_\infty = W \otimes \mathbf{R}$, then $C_0(V_\infty) = \operatorname{End}(W_\infty)$.

Now, suppose that $\operatorname{rank}(L) = \dim(V) = 2n$. By essentially the same argument as above, we find that there is a rational vector space W such that there is an isomorphism $\rho\colon C(V) \to \operatorname{End}(W)$. Let $\{x_1, \ldots, x_l\}$ be an orthogonal basis for $V = \mathbf{Q}^l$, which we may choose so that $q(x_1)\cdots q(x_l) = 1$, and let $z = x_1 \cdots x_l$. For each i,

$$\begin{aligned} x_i z &= (-1)^{i-1} x_{\sigma_i} \\ z x_i &= (-1)^{l-i} x_{\sigma_i}, \end{aligned}$$

where $\sigma_i = \{1, \ldots, l\} \setminus \{i\}$. Therefore z anticommutes with every element of V and is in the center of $C_0(V)$. Moreover,

$$z^2 = x_1 \cdots x_l x_1 \cdots x_l = (-1)^{l(l-1)/2} x_1^2 \cdots x_l^2 = 1 \tag{28}$$

since $l \equiv 0 \pmod 8$. Let $\zeta_0 = \frac{1}{2} + \frac{1}{2}z$ and $\zeta_1 = \frac{1}{2} - \frac{1}{2}z$. We find that ζ_0 and ζ_1 are central in $C_0(V)$, and that

$$\begin{aligned} \zeta_0^2 &= \zeta_0, \\ \zeta_1^2 &= \zeta_1, \\ \zeta_0\zeta_1 &= 0, \\ \zeta_0 + \zeta_1 &= 1. \end{aligned} \tag{29}$$

Therefore, $\zeta_0 C_0(V)$ and $\zeta_1 C_0(V)$ are two-sided ideals of $C_0(V)$ and $C_0(V)$ is the direct sum of $\zeta_0 C_0(V)$ and $\zeta_1 C_0(V)$. Furthermore, since $\rho(\zeta_0)$ and $\rho(\zeta_1)$ are idempotent elements of $\mathrm{End}(W)$, there exist subspaces W_0 and W_1 of W such that $\rho(\zeta_i)$ is a projection onto W_i. it follows that $W = W_0 \oplus W_1$, $C_0(V) = \mathrm{End}(W_0) \oplus \mathrm{End}(W_1)$, and the representation ρ decomposes into two irreducible representations $\rho^+ \colon C_0(V) \to \mathrm{End}(W_0)$ and $\rho^- \colon C_0(V) \to \mathrm{End}(W_1)$. Tensoring with $\mathbf{Q}_p$ demonstrates that $\dim(W_0) = \dim(W_1) = 2^{n-1}$.

Connecting the half-spin representations constructed above to their local counterparts is a slightly delicate business. In general, we know that $W_i \otimes \mathbf{Q}_p = W_{j,p}$, but we have no guarantee that $i = j$. Let us, by replacing z by $-z$ if necessary, at least let $W_0 \otimes \mathbf{Q}_2 = W_{0,2}$. We claim that by adjusting our choice of isotropic sublattice N_p, we may force $W_0 \otimes \mathbf{Q}_p$ to be $W_{0,p}$ for each odd prime p. First of all, we note that the only elements of the center of $C_0(V_p)$ that have square 1 are ± 1 and $\pm z$. (Recall that the center of $C_0(V_p)$ has dimension 2 by Corollary 3.) We may choose an orthogonal basis $\{y_1, \ldots, y_l\}$ for L_p such that $q(y_1) \cdots q(y_l) = 1$ since p is odd and assume without loss of generality that $y_1 \cdots y_l = z$ since $y_1 \cdots y_l$ is central in $C_0(V_p)$ and linearly independent of 1. Note that since $q(y_1) \in \mathbf{Z}_p^*$, the reflection r_{y_1} is in the orthogonal group of L_p. Now, if it happens that $\rho_p^+(C_0(V_p))$ fixes $W_{1,p}$ instead of $W_{0,p}$, we may redefine the local half-spin representations by replacing N_p with $r_{y_1} N_p$, which has the effect of interchanging z and $-z$ in $C_0(V_p)$ and thereby interchanging ρ_p^+ and ρ_p^-. For $i = 0, 1$, we set $W_{i,\infty} = W_i \otimes \mathbf{R}$.

In many circumstances, we may globally identify the even and odd half-spin representations. If u is an invertible element of $C_1(V)$, then the fact that z anticommutes with u implies that

$$u\zeta_0 = \zeta_1 u. \tag{30}$$

We conclude that

$$\rho(u)W_0 = \rho(u)\rho(\zeta_0)W = \rho(\zeta_1)\rho(u)W = \rho(\zeta_1)W = W_1, \tag{31}$$

and therefore $\rho(u)|_{W_0}$ is a vector space isomorphism of W_0 and W_1 with inverse $\rho(u^{-1})|_{W_1}$. When $u \in C^*(V)$, we will find that $\rho(u)$ also preserves the invariant quadratic structure on W_0 and W_1 described below.

In addition to the global spin representations, we may also produce inner products invariant under the actions of $\mathrm{Spin}(V)$ and of $\mathfrak{so}(V)$.

Lemma 19 *Suppose that G is a group and $\tau\colon G \to \mathrm{End}(X)$ is a representation of G, where X is a finite-dimensional vector space over $\mathbf{Q}$. If X^G is the subspace of vectors fixed by the action of G, then*

$$X^G \otimes \mathbf{R} = (X \otimes \mathbf{R})^G.$$

Proof. Since X^G has finite codimension in X, there exist group elements $g_1, \ldots, g_k$ such that X^G is the kernel of the map from X to $X \oplus \cdots \oplus X$ given by

$$x \mapsto (x - \tau(g_1)x, \ldots, x - \tau(g_k)x).$$

The functor $\cdot \otimes_{\mathbf{Q}} \mathbf{R}$ from $\mathbf{Q}$-vector spaces to $\mathbf{R}$-vector spaces is exact, and $(X \otimes \mathbf{R})^G$ is clearly contained in the kernel of the corresponding map from $X \otimes \mathbf{R}$ to $(X \otimes \mathbf{R}) \oplus \cdots \oplus (X \otimes \mathbf{R})$. Therefore $(X \otimes \mathbf{R})^G \subseteq X^G \otimes \mathbf{R}$. Conversely, any element of $X^G \otimes \mathbf{R}$ is fixed by the action of G on $X \otimes \mathbf{R}$. □

Proposition 20 *The representation space W for $C_0(V)$ may be equipped with a positive definite, symmetric bilinear form β which is invariant under $\rho(\mathrm{Spin}(V))$. When V has odd dimension, β is unique up to scaling. When V has even dimension, we obtain by restriction positive definite, symmetric bilinear forms β_0 on W_0 and β_1 on W_1 invariant under $\rho^+(\mathrm{Spin}(V))$ and under $\rho^-(\mathrm{Spin}(V))$ respectively, and β_0 and β_1 are unique up to scaling.*

Proof. We present the argument for the odd-dimensional case; the even-dimensional case is proven in more or less the same manner. A symmetric bilinear form on W corresponds to an element of the space $\mathrm{Sym}^2(W)^*$, and the set of symmetric bilinear forms on W invariant under the action of $\mathrm{Spin}(V)$ is the subspace $(\mathrm{Sym}^2(W)^*)^{\mathrm{Spin}(V)}$ of elements that are fixed by the action of $\mathrm{Spin}(V)$ on $\mathrm{Sym}^2(W)^*$. By Lemma 19,

$$\begin{aligned}(\mathrm{Sym}^2(W)^*)^{\mathrm{Spin}(V)} \otimes \mathbf{R} &= (\mathrm{Sym}^2(W)^* \otimes \mathbf{R})^{\mathrm{Spin}(V)} \\ &= (\mathrm{Sym}^2(W_\infty)^*)^{\mathrm{Spin}(V)}.\end{aligned}$$

The standard argument by integration over the compact group $\mathrm{Spin}(V_\infty)$ exhibits the existence of a positive definite invariant bilinear form on W_∞. Since the spin representation of W_∞ is absolutely irreducible, $(\mathrm{Sym}^2(W_\infty)^*)^{\mathrm{Spin}(V)}$ is a vector space of dimension 1, and its positive definite elements form an open subset. Therefore $(\mathrm{Sym}^2(W)^*)^{\mathrm{Spin}(V)}$ has dimension 1 and contains some positive definite element, which is the invariant, positive definite, symmetric bilinear form that we seek. □

Proposition 21 *If* rank(L) *is odd, then there is a lattice $M \subset W$ that is stabilized by $\rho(C_0(L))$, and this lattice is unique up to scaling by $\mathbf{Q}^*$. If* rank(L) *is even, then there is a lattice $M \subset W$ that is stabilized by $\rho(C(L))$, there is a lattice $M_0 \subset W_0$ that is stabilized by $\rho^+(C_0(L))$, and there is a lattice $M_1 \subset W_1$ that is stabilized by $\rho^-(C_0(L))$, and these lattices are unique up to scaling by $\mathbf{Q}^*$.*

Proof. As the arguments in both cases are essentially the same, we give only the odd rank case. Let $\phi_1, \ldots, \phi_k$ be a $\mathbf{Z}$-basis for $\rho(C_0(L))$, and let M' be any lattice in W. Let

$$M = \phi_1(M') + \cdots + \phi_k(M').$$

M is a finitely generated $\mathbf{Z}$-submodule of W, and since ρ is an irreducible representation, $M \otimes \mathbf{Q} = W$. Thus, M is a lattice. A routine calculation shows that since $\phi_1, \ldots, \phi_k$ is a basis of $\rho(C_0(L))$, M is stable under $\rho(C_0(L))$.

Suppose that M'' is another stable lattice for $C_0(L)$. By Proposition 18, $M'' \otimes \mathbf{Z}_p = p^{i_p}(M \otimes \mathbf{Z}_p)$, where $i_p = 0$ for all but finitely many p. If we set $a = \prod p^{i_p}$, then $M'' = aM$. □

Remark. It follows from the local equivalence that when rank(L) is odd, $C_0(L)$ is the full endomorphism ring End(M), and that when rank(L) is even, $C_0(L) = \mathrm{End}(M_0) \oplus \mathrm{End}(M_1)$.

When we say that M is **stable** under the action of $C_0(L)$, we mean only that $\rho(C_0(L))$ maps M into M, and not that its quadratic structure is in any way preserved. By contrast, under the representations of Spin(L) and $\mathfrak{spin}(L)$, the invariant inner product that we exhibited in Proposition 20 is preserved (either in the Lie group sense or the Lie algebra sense), and accordingly, M is **invariant** under these representations. With the proper scaling, we find that M, M_0 and M_1 are actually even unimodular lattices.

Theorem 22 *If L has odd rank, then there is a unique positive definite, even, unimodular lattice M that is invariant under the spin representations of* Spin(L) *and* $\mathfrak{spin}(L)$. *If L has even rank, then there is a unique positive definite even, unimodular lattice M_0 that is invariant under the even half-spin representations of* Spin(L) *and* $\mathfrak{spin}(L)$ *and a unique positive definite, even, unimodular lattice M_1 that is invariant under the odd half-spin representations of* Spin(L) *and* $\mathfrak{spin}(L)$.

Proof. Suppose that the rank of L is odd. Let β be an invariant, positive definite, symmetric bilinear form on W, and let M be a lattice in W stabilized by the action of $C_0(L)$ as in Proposition 21. We may imbed M^* in W as

$$M^* = \{x \in W \mid \beta(x, M) \subseteq \mathbf{Z}\}.$$

Since β is invariant under the action of $\mathfrak{spin}(L)$, we find that if $u \in \mathfrak{spin}(L)$,

$$\beta(\rho(u)M^*, M) = -\beta(M^*, \rho(u)M) \subseteq -\beta(M^*, M) \subseteq \mathbf{Z},$$

and therefore that M^* is stabilized by $\mathfrak{spin}(L)$. By Propositions 17 and 21, there is a unique $a \in \mathbf{Q}_+^*$ such that $M^* = aM$. Let $\beta' = a\beta$. If $\{x_1, \ldots, x_m\}$ is a basis for M and $\{x_1^*, \ldots, x_m^*\}$ is the corresponding dual basis with respect to β, the fact that

$$\beta'(a^{-1}x_i^*, x_j) = \beta(x_i^*, x_j) = \delta_{i,j}$$

implies that the lattice M is integral and unimodular under the bilinear form β'. Suppose that M is not even. In this case, it has an even sublattice M' of index 2, and this sublattice must be invariant under the action of $\mathfrak{spin}(L)$. But since $\operatorname{rank}(M) > 1$, this clearly violates Proposition 17. Therefore M is even.

Now suppose that the rank of L is even, and choose a lattice M in W stabilized by the action of $C(L)$. By applying a similar argument to that given above to the sublattices $M_0 = M \cap W_0$ and $M_1 = M \cap W_1$, we may conclude that M_0 and M_1 each carry a uniquely defined invariant structure of a positive definite, even, unimodular lattice. □

Remark. By Proposition 17, any lattice that is invariant under $\mathfrak{spin}(L)$ will be invariant under $\mathrm{Spin}(L)$. However, the converse may not be true, as demonstrated by the existence of even unimodular lattices whose special orthogonal groups are $\{\pm 1\}$.

Suppose that L has even rank and $\mathrm{SO}(L) \neq \mathrm{O}(L)$. If $u \in \mathrm{O}(L) \setminus \mathrm{SO}(L)$, then a refinement of the computation in Equation 31 shows that $\rho(u)$ gives a $\mathbf{Z}$-module isomorphism from M_0 to M_1. We also find that $\rho(u)$ sends the inner product on M_0 to a $\rho^-(\mathfrak{spin}(L))$-invariant, unimodular inner product on M_1, and thus that ρ is a lattice isomorphism.

We now summarize our spin representation construction. If L is an even lattice of determinant 2 in rank $8k-1$, the spin representation of L yields an even unimodular lattice of rank 2^{4k-1}. If L is an even lattice of determinant 2 in rank $8k+1$, the spin representation of L yields an even unimodular lattice of rank 2^{4k}. If L is an even unimodular lattice in rank $8k$, each half-spin representation yields an even unimodular lattice of rank 2^{4k-1}; when $\mathrm{SO}(L) = \mathrm{O}(L)$, as for the Leech lattice, these half-spin lattices might be non-isomorphic.

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