Math 322

Explorations of Ideals and Quotient Rings

1. Suppose that R is a commutative ring with unity.

   (a) Prove that if $a, b \in R$, then b divides a in R if and only if $aR \subseteq bR$.

   Hint. There is a shortcut for proving that $aR \subseteq I$ when I is an ideal of R. I recommend using it.

   (b) Prove that if $a, b \in R \setminus \{0\}$, then a and b are associates in R if and only if $aR = bR$.

   Recall that a **and** b **are associates in** R means that $a = ub$ for some $u \in R^*$.

2. Suppose that R is a non-trivial commutative ring with unity. Prove that R is a field if and only if the only ideals of R are $\{0\}$ and R.

3. Remember how we looked at that four element "ring" E and I said *trust me, it's a ring*? (Probably not: E was that alleged ring that we made by starting with $\mathbf{Z}_2$ and throwing in the imaginary roots α and β of the polynomial $x^2 + x + \bar{1}$.)

   Now we're going to prove that it *is* a ring.

   (a) Let $I = (x^2+x+\bar{1})\mathbf{Z}_2[x]$, which is an ideal in $\mathbf{Z}_2[x]$, and let $F = \mathbf{Z}_2[x]/I$. Prove that F contains exactly four elements, which we will denote by $Z = \bar{0} + I$, $Y = \bar{1} + I$, $A = x + I$, and $B = (x + \bar{1}) + I$.

   (b) Prove that $\{Z, Y\}$ is a subring of F isomorphic to $\mathbf{Z}_2$.

   (c) Prove that A and B are roots of the polynomial $Yx^2 + Yx + Y$ in $F[x]$.

   (d) Conclude that F is isomorphic to E. See, I told you so!

4. And now for something completely different– questions with actual numbers!

   (a) What is the minimal polynomial of $2 + \sqrt[3]{2}$ over $\mathbf{Q}$?

   Hint. Try some old-fashioned high-school algebra.

   (b) What is the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbf{Q}$?

   Hint. Ditto. This time, proving minimality will be more challenging.