Math 322
Explorations of Ideals and Kernels
(+ one odd end)

1. Recall that $\mathcal{F}(\mathbf{R})$ is the ring of functions from $\mathbf{R}$ to $\mathbf{R}$ under the operations of function addition and function multiplication.

   Prove that $\mathcal{F}(\mathbf{R})$ is not an integral domain.

2. If R is a commutative ring and $a \in R$, prove that the set $aR = \{ar : r \in R\}$ is an ideal in R. (In fact, by the absorption property, if R is a ring with unity, aR is the smallest ideal of R containing a.)

3. We will now work in $\mathbf{R}[x]$ and in $\mathbf{C}$.

   (a) Let $g(x)$ be the least-degree monic polynomial in $\mathbf{R}[x]$ that has $2-i$ as a root. What is $g(x)$?

   Hint. Do you know anything else in $\mathbf{C}$ that must be a root of $g(x)$? Does that help?

   (b) Prove that if $f(x) \in \mathbf{R}[x]$ has $2-i$ as a root, then $g(x)|f(x)$.

4. The Fundamental Homorphism Theorem revisited.

   (a) Suppose $\phi: R \longrightarrow R'$ is a ring homomorphism, and hence also an additive group homomorphism. In group theory, we proved that $(R/\ker(\phi), +) \cong (\phi(R), +)$ via the group isomorphism $\overline{\phi}$ defined by

   $$\overline{\phi}(a + \ker(\phi)) = \phi(a).$$

   Prove that $\overline{\phi}$ is in fact a ring homomorphism, so that $R/\ker(\phi) \cong \phi(R)$ as a ring.

   As in group theory, this is very useful for computing quotient rings.

   (b) Prove that $\mathbf{R}[x]/(x^2+1)\mathbf{R}[x] \cong \mathbf{C}$.

   Hint. Consider the evaluation homomorphism $\phi_i: \mathbf{R}[x] \longrightarrow \mathbf{C}$. What is its kernel? What is its image?