Math 322

Explorations of Ring Axioms

The following are all examples of rings: $\mathbf{Z}$, $\mathbf{Z}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$, $2\mathbf{Z}$, $\mathbf{R}$, and $M_2(\mathbf{R})$, where

$$M_2(\mathbf{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbf{R} \right\}.$$

Each of these sets has a binary operation $+$ (called addition) and a second binary operation $\cdot$ (called multiplication).

1. What properties do the sets $\mathbf{Z}$, $\mathbf{Z}_5$, $2\mathbf{Z}$, $\mathbf{R}$, $M_2(\mathbf{R})$, and their respective addition operations have? How do they compare to the group axioms?

2. What properties do the sets $\mathbf{Z}$, $\mathbf{Z}_5$, $2\mathbf{Z}$, $\mathbf{R}$, $M_2(\mathbf{R})$, and their respective multiplication operations have? How do they compare to the group axioms?

3. So far, we have only considered addition and multiplication in isolation. Are there any rules of arithmetic in $\mathbf{Z}$, $\mathbf{Z}_5$, $2\mathbf{Z}$, $\mathbf{R}$, and $M_2(\mathbf{R})$ that involve both of the operations at once?

4. What do you think the axioms that define a ring are?