Math 151 Sample Final Exam

Numerical Solutions

1. (12 points)

   Find the following limits.

   (a) $\lim_{x\to 1} \frac{x-1}{x^2-7x+6} = \lim_{x\to 1} \frac{1}{x-6} = -\frac{1}{5}$

   (b) $\lim_{x\to 2} \frac{x-1}{x^2-5x+6}$ does not exist

   (c) $\lim_{x\to -\infty} \frac{x^2-27x}{5-4x^2} = -\frac{1}{4}$

2. (6 points)

   Using the definition of the derivative, find the derivative of $g(x) = \sqrt{2x+1}$.

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \\
&= \lim_{h\to 0} \frac{2(x+h)+1-(2x+1)}{h(\sqrt{2(x+h)+1}+\sqrt{2x+1})} \\
&= \lim_{h\to 0} \frac{2h}{h(\sqrt{2(x+h)+1}+\sqrt{2x+1})} \\
&= \lim_{h\to 0} \frac{2}{(\sqrt{2(x+h)+1}+\sqrt{2x+1})} \\
&= \frac{2}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}
\end{aligned}
$$

3. (20 points)

   Find the following derivatives.

   (a) $f(x) = x^3 + x^2 - 5\sqrt{x}$; find $f''(x)$.
   $f'(x) = 3x^2 + 2x - (5/2)x^{-1/2}$
   $f''(x) = 6x + 2 + (5/4)x^{-3/2}$

   (b) $g(t) = \sec(\pi - t^2)$; find $g'(t)$.
   $g'(t) = \sec(\pi - t^2)\tan(\pi - t^2)(-2t)$

   (c) $y = \frac{w+9}{w^2-3}$; find $\frac{dy}{dw}$.
   $\frac{dy}{dw} = \frac{w^2-3-(w+9)(2w)}{(w^2-3)^2}$

   (d) $G(x) = \int_3^x \tan\sqrt{t}\, dt$; find $G'(x)$.
   $G'(x) = \tan\sqrt{x}$

4. (6 points)

   Find the tangent line to the curve $x^3 - 6xy + y^2 = 2x - 7$ at the point $(2, 1)$.

   $3x^2 - 6(x\frac{dy}{dx} + y) + 2y\frac{dy}{dx} = 2$

   $12 - 6(2\frac{dy}{dx} + 1) + 2\frac{dy}{dx} = 2$

   $\frac{dy}{dx} = 2/5$

   Tangent line: $y - 1 = (2/5)(x - 2)$

5. (8 points)

   Find the maximum and minimum values of $f(x) = x^4 - 2x^3$ on the interval $[-1, 4]$.

   $f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3)$

x	$f(x)$
-1	3
0	0
$3/2$	$-27/16$
4	128

   Minimum $= -27/16$, Maximum $= 128$

6. (16 points)

   Compute the following.

   (a) $\int_0^3 (x^2 - 1); dx = \left[x^3/3 - x\right]_0^3 = 6$

   (b) $\int \sec^2(5t)\ dt = (1/5)\tan(5t) + C$

   (c) The Riemann sum for $f(x) = x - x^2$ on the interval $[-2, 4]$ with three pieces and right endpoints.

   $f(0) \cdot 2 + f(2) \cdot 2 + f(4) \cdot 2 = -28$

   (d) The area between $y = 6x - x^2$ and the x-axis.

   Hint. To figure out where to start and end your integral, you might find drawing a graph helpful.

   Area $= \int_0^6 (6x - x^2)\ dx = \left[3x^2 - x^3/3\right]_0^6 = 36$

7. (12 points)

   Susan plans to build a box with a square bottom, an open top, and a capacity of 1000 in^3. The material for the sides costs $1 per square inch, but the bottom will be made of sturdier material that costs $2 per square inch.

   What dimensions should Susan give the box so that she can build it as cheaply as possible?

   Let the side of the square base be b and the height of the box be h.

   Volume $= 1000 = b^2h$, so $h = 1000/b^2$

We want to minimize the cost $C = 1 \cdot 4 \cdot bh + 2 \cdot b^2 = 4000/b + 2b^2$.

$C'(b) = -4000/b^2 + 4b$

$C'(0)$ is undefined, but b must be positive, so that is outside the interval of allowed values.

$C'(b) = 0$ when $b = 10$.

Since $C'(5)$ is negative and $C'(20)$ is positive, the cost is a minimum when $b = 10$.

$h = 1000/10^2 = 10$

The cheapest box has a base with 10 in sides and a height of 10 in.

8. (12 points)

   A clown is shot out of a cannon. Her vertical position in feet as a function of time in seconds is

$$c(t) = 96t - 16t^2.$$

   (a) What is the clown's upward velocity as a function of time?

   $v(t) = c'(t) = 96 - 32t$ ft/sec

   (b) When is the clown's upward velocity 0 ft/sec?

   $v(t) = 96 - 32t = 0$ when $t = 3$ seconds

   (c) How high does the clown fly?

   $c(3) = 144$ ft

   (d) The clown is aimed to fall into a net 80 feet above the ground. How fast is the clown falling when she is about to hit the net?

   The clown hits the net when $c(t) = 96t - 16t^2 = 80$, which is when $16t^2 - 96t + 80 = 16(t-1)(t-5) = 0$. At $t = 1$, the clown is rising past the net, so she falls into the net at $t = 5$ seconds.

   $v(5) = 96 - 32 \cdot 5 = -64$ ft/sec, so the clown is falling at 64 ft/sec

9. (8 points)

   (a) State the Mean Value Theorem.

   If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   (b) For $f(x) = \frac{1}{x}$ on the interval $[1, 9]$, give the number c guaranteed by the Mean Value Theorem.

   $\dfrac{f(9) - f(1)}{9 - 1} = \dfrac{1/9 - 1}{8} = -(1/9)$

   $f'(c) = -(1/c^2) = -(1/9)$

   $c = 3$