

Graph the following functions. For each one, you will first extract the following information.

- From the original function: What are the horizontal and vertical asymptotes (if any)?
- From the first derivative: Where is the original function increasing and where is it decreasing? What are the local maxima and minima?
- From the second derivative: Where is the original function concave up and where is it concave down? What are the inflection points?

1. $f(x) = x^4 - 8x^2 + 4$

f is a polynomial, so it has no vertical or horizontal asymptotes.

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x + 2)(x - 2)$$

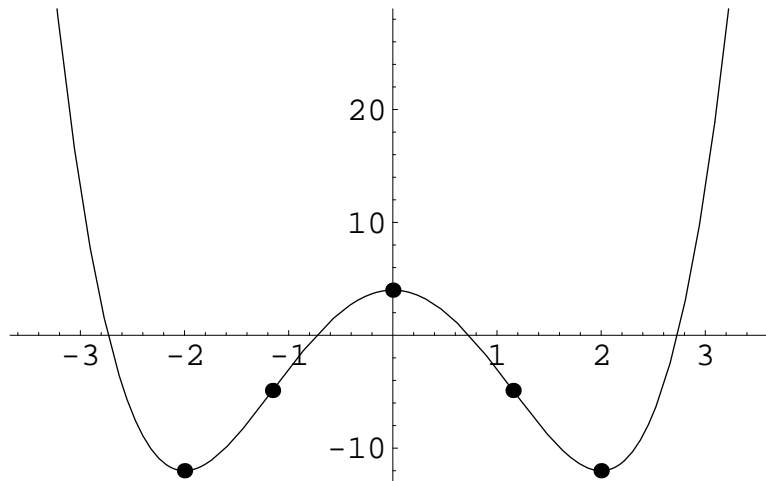
Critical points (where $f'(x)$ could change sign): $-2, 0, 2$

f is decreasing on $(-\infty, -2)$ and on $(0, 2)$ and increasing on $(-2, 0)$ and $(2, \infty)$. This means that f has local minima at $x = \pm 2$ and a local maximum at $x = 0$. The local minima are $f(-2) = f(2) = -12$, and the local maximum is $f(0) = 4$.

$$f''(x) = 12x^2 - 16 = 4(3x^2 - 4)$$

Where $f''(x)$ could change sign: $-\sqrt{4/3}, \sqrt{4/3}$

f is concave up on $(-\infty, -\sqrt{4/3})$ and on $(\sqrt{4/3}, \infty)$ and concave down on $(-\sqrt{4/3}, \sqrt{4/3})$. This means that f has inflection points $(-\sqrt{4/3}, -24/9)$ and $(\sqrt{4/3}, -24/9)$.



2. $g(x) = \frac{x-1}{x^2}$

The denominator of $g(x)$ is 0 when $x = 0$, so g has the vertical asymptote $x = 0$.

$$\lim_{x \rightarrow \infty} \frac{x-1}{x^2} = 0 \text{ and } \lim_{x \rightarrow -\infty} \frac{x-1}{x^2} = 0,$$

which means that g has the horizontal asymptote $y = 0$.

$$g'(x) = \frac{x^2 - (x-1)(2x)}{x^4} = \frac{-x^2 + 2x}{x^4} = \frac{2-x}{x^3}$$

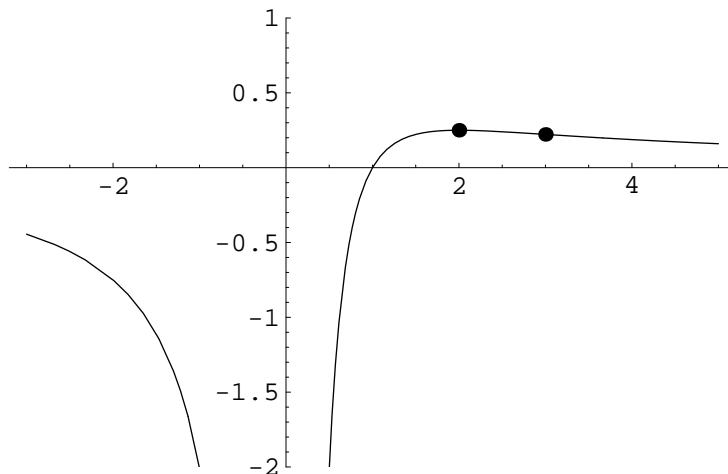
Critical numbers (where $g'(x)$ could change sign): 0, 2

g is decreasing on $(-\infty, 0)$ and on $(2, \infty)$ and g is increasing on $(0, 2)$. This means that g has a local maximum of $g(2) = 1/4$ at $x = 2$. (There is no local extreme at $x = 0$ because of the asymptote.)

$$g''(x) = \frac{x^3(-1) - (2-x)(3x^2)}{x^6} = \frac{2x^3 - 6x^2}{x^6} = \frac{2(x-3)}{x^4}$$

Where $g''(x)$ could change sign: 0, 3

g is concave down on $(-\infty, 0)$ and on $(0, 3)$ and concave up on $(3, \infty)$. This means that g has one inflection point, $(3, 2/9)$.



3. $h(x) = (x^2 - 4)^{2/3}$

$h(x)$ is defined everywhere, so h has no vertical asymptotes.

$$\lim_{x \rightarrow \infty} (x^2 - 4)^{2/3} = \infty \text{ and } \lim_{x \rightarrow -\infty} (x^2 - 4)^{2/3} = \infty,$$

so h doesn't have any horizontal asymptotes either.

$$h'(x) = 2x(2/3)(x^2 - 4)^{-1/3} = \frac{4x}{3\sqrt[3]{x^2 - 4}}$$

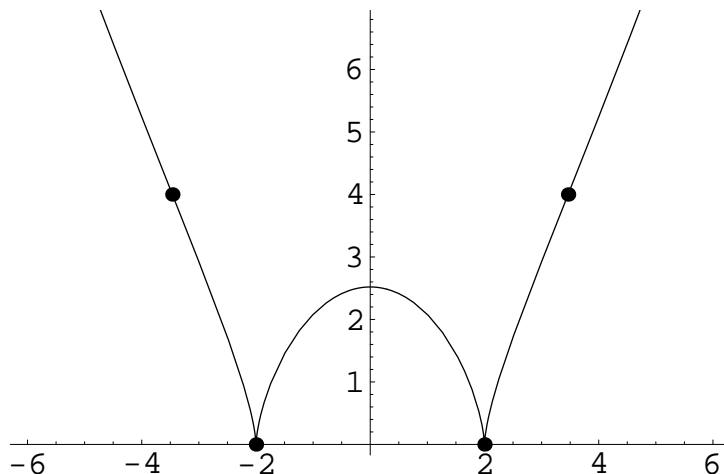
Critical numbers (where $h'(x)$ could change sign): $-2, 0, 2$

h is decreasing on $(-\infty, -2)$ and on $(0, 2)$ and increasing on $(-2, 0)$ and on $(2, \infty)$. This means that h has local minima at $x = \pm 2$ and a local maximum at $x = 0$; both local minima are $h(2) = h(-2) = 0$, and the local maximum is $h(0) = 4^{2/3}$.

$$h''(x) = \frac{4}{3} \left(\frac{(x^2 - 4)^{1/3} - x(1/3)(x^2 - 4)^{-2/3}(2x)}{(x^2 - 4)^{2/3}} \right) = \frac{4}{3} \left(\frac{(x^2 - 4) - (2/3)x^2}{(x^2 - 4)^{4/3}} \right) = \frac{4}{3} \left(\frac{(1/3)x^2 - 4}{(x^2 - 4)^{4/3}} \right)$$

Where $h''(x)$ could change sign: $-2\sqrt{3}$, -2 , 2 , $2\sqrt{3}$

h is concave up on $(-\infty, -2\sqrt{3})$ and on $(2\sqrt{3}, \infty)$ and concave down on $-2\sqrt{3}, -2$, on $(-2, 2)$ and on $(2, 2\sqrt{3})$. (h is not concave down at ± 2 because of the sharp corners, since $h'(-2)$ and $h'(2)$ are undefined.) This means that h has inflection points $(-2\sqrt{3}, 4)$ and $(2\sqrt{3}, 4)$.



The change of concavity at the marked inflection points is hard to see in the computer picture because at the scale of the graph, the shift is very subtle.