Math 151
Exploring Applications of Derivatives
Numerical Solutions

1. Verify that the function $f(x) = \sqrt{x-3}$ on the interval $[4, 28]$ satisfies the conditions for the Mean Value Theorem, and find the number $c$ in $[4, 28]$ guaranteed by the Theorem.

   $f(x)$ is continuous on $[4, 28]$ because for $x$ in $[4, 28]$, $x - 3$ is positive, and since $f'(x) = 1/2\sqrt{x-3}$, $f(x)$ is also differentiable on $(4, 28)$.

   The number $c$ for which
   $$f'(c) = \frac{f(b) - f(a)}{b - a} = 1/6$$
   is $c = 12$.

2. Caitlin plans to build a box with a square bottom and an open top. The material for the sides costs 2¢ per square inch, but the bottom will be made of sturdier material that costs 3¢ per square inch. She has \$9.00 budgeted for the box materials.

   What is the largest volume Caitlin can give her box?

   *Hint.* You will probably be happier if you measure the cost in cents, as opposed to dollars.

   Let the base of the box have side length $w$ and let the height of the box be $h$.

   Total cost (in cents) $= 3w^2 + 8wh = 900$, so
   $$h = \frac{900 - 3w^2}{8w}.$$

   We are trying to maximize the volume, which is
   $$V = w^2 h = w^2 \left( \frac{900 - 3w^2}{8w} \right) = \frac{w(900 - 3w^2)}{8} = (900/8)w - (3/8)w^3.$$

   $V' = 900/8 - (9/8)w^2$, so the critical numbers are $w = \pm 10$. Since $w$ is a measurement of length, $w$ can't be negative, so the only critical number that matters is 10.

   $V'(0) = 900/8 > 0$, and $V'(20) = -2700/8 < 0$, which means that $V$ is increasing before $w = 10$ and decreasing after $w = 10$.

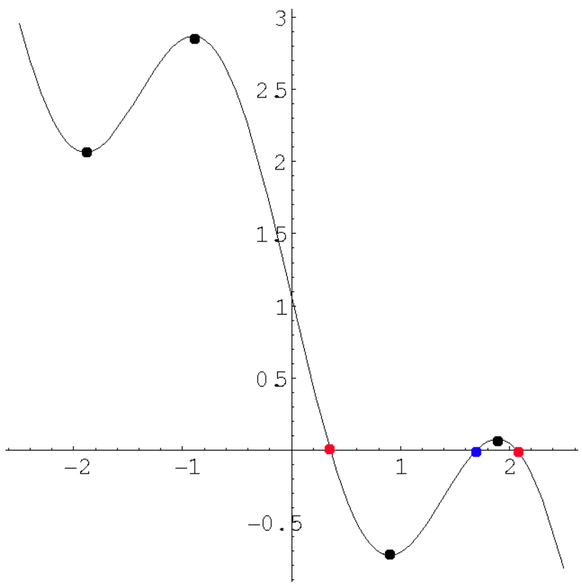
   Therefore, the largest possible volume is
   $$V(10) = 750 \text{ in}^3.$$

3. A local television station is running a *Lost* marathon, and Chris's Candy Store is buying ad time during the marathon. Market research shows that every time Chris airs his ad, he gains 1,000 new customers. However, once the ad airs $t$ times, a total of $20t^2$ of these potential customers have become so sick of the ad that they vow never to shop at Chris's. How many ads should Chris run to get the most new customers?

   Number of new customers after the commerical airs $t$ times: $N(t) = 1000t - 20t^2$.

   $N'(t) = 1000 - 40t$, so the only critical number is 25. Since $N''(t) = -40$, the second derivative test says that since $N$ is concave down, it has a maximum at 25. (You can look at the first derivative if you prefer.)

   Chris should air the ad 25 times.




4. Above is the graph of $f'(x)$. At what values of $x$ does $f(x)$ have a local maximum? A local minimum? An inflection point?

   Note that the graph here is NOT the graph of $f(x)$. So, for example, it is not true that $f(x)$ has a local maximum at $x = -0.8$.

   At the $x$ values of the red points, $f(x)$ has a maximum. At the $x$ value of the blue point, $f(x)$ has a minimum. At the $x$ values for the black points, $f(x)$ has an inflection point.

5. Let $g(x) = 2 - \frac{6}{x} + \frac{6}{x^2} = \frac{2x^2-6x+6}{x^2}$.

   (a) Find all the asymptotes of the graph $y = f(x)$.
   The graph has a vertical asymptote when the denominator is 0, which is $x = 0$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(2 - \frac{6}{x} + \frac{6}{x^2}\right) = 2$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(2 - \frac{6}{x} + \frac{6}{x^2}\right) = 2$$

   So the graph has a horizontal asymptote $y = 2$.

   (b) Find the intervals on which $f$ is increasing and on which $f$ is decreasing.
   $f'(x) = \frac{6}{x^2} - \frac{12}{x^3} = \frac{6-12x}{x^3}$
   $f'(x)$ is undefined at $x = 0$ and $f'(x) = 0$ at $x = 2$.

| $x$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $f'(x)$ | 18 | | $-6$ | 0 | $\frac{2}{9}$ |
| sign of $f'(x)$ | + | | $-$ | | + |

   Thus $f$ is increasing on $(-\infty, 0)$ and on $(2, \infty)$ and decreasing on $(0, 2)$.

   (c) Find the intervals on which $f$ is concave up and on which $f$ is concave down.
   $f''(x) = \frac{-12}{x^3} + \frac{36}{x^4} = \frac{-12x+36}{x^4}$

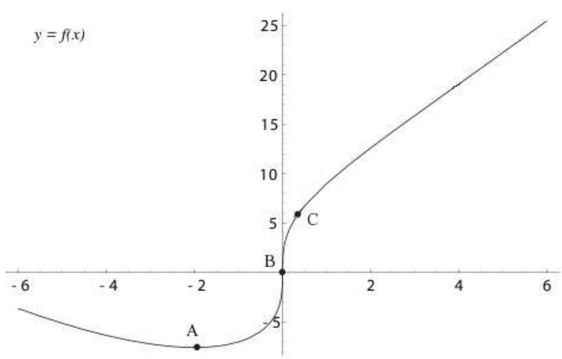
$f''(x)$ is undefined at $x = 0$ and $f''(x) = 0$ at $x = 3$.

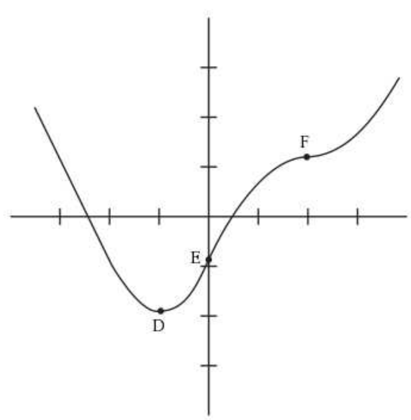
| $x$ | $-1$ | $0$ | $1$ | $3$ | $4$ |
|---|---|---|---|---|---|
| $f''(x)$ | 48 | | 24 | 0 | $\frac{-3}{64}$ |
| sign of $f'(x)$ | + | | + | | $-$ |

Thus $f$ is concave up on $(-\infty, 0)$ and $(0, 3)$ and concave down on $(3, \infty)$

(d) Sketch the graph $y = f(x)$. Be sure to label all asymptotes, local maxima, local minima, and inflection points.

6. Below are the graphs of two functions.




At points A, B, and C, is $f'(x)$ positive, negative, zero, or undefined? Is $f''(x)$ positive, negative, zero, or undefined?

At points D, E, and F, is $g'(x)$ positive, negative, zero, or undefined? Is $g''(x)$ positive, negative, zero, or undefined?

| | $f'(x)$ | $f''(x)$ |
|---|---|---|
| A | 0 | + |
| B | undefined | undefined |
| C | + | - |

| | $g'(x)$ | $g''(x)$ |
|---|---|---|
| D | 0 | + |
| E | + | 0 |
| F | 0 | 0 |