

# Putnam Seminar — Problems 8

## PUTNAM PROBLEMS

**Instructions.** All Putnam problems. All doable.

2005 A1 Show that every positive integer is a sum of one or more numbers of the form  $2^r 3^s$ , where  $r$  and  $s$  are nonnegative integers and no summand divides another. (For example,  $23 = 9 + 8 + 6$ .)

2005 B1 Find a nonzero polynomial  $P(x, y)$  such that  $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$  for all real numbers  $a$ . (Note:  $\lfloor \nu \rfloor$  is the greatest integer less than or equal to  $\nu$ .)

2005 B2 Find all positive integers  $n, k_1, \dots, k_n$  such that  $k_1 + \dots + k_n = 5n - 4$  and

$$\frac{1}{k_1} + \dots + \frac{1}{k_n} = 1.$$

2004 A1 Basketball star Shanille O'Keal's team statistician keeps track of the number,  $S(N)$ , of successful free throws she has made in her first  $N$  attempts of the season. Early in the season,  $S(N)$  was less than 80% of  $N$ , but by the end of the season,  $S(N)$  was more than 80% of  $N$ . Was there necessarily a moment in between when  $S(N)$  was exactly 80% of  $N$ ?

2004 A3 Define a sequence  $\{u_n\}_{n=0}^{\infty}$  by  $u_0 = u_1 = u_2 = 1$ , and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all  $n \geq 0$ . Show that  $u_n$  is an integer for all  $n$ . (By convention,  $0! = 1$ .)

2004 B1 Let  $P(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_0$  be a polynomial with integer coefficients. Suppose that  $r$  is a rational number such that  $P(r) = 0$ . Show that the  $n$  numbers

$$\begin{aligned} &c_n r, c_n r^2 + c_{n-1} r, c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \\ &\dots, c_n r^n + c_{n-1} r^{n-1} + \dots + c_1 r \end{aligned}$$

are integers.