Putnam Seminar — Problems 7

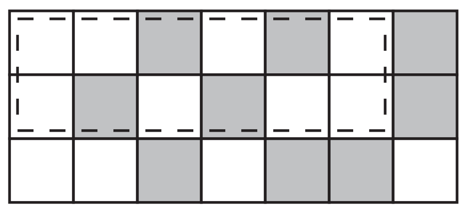
Pigeonhole Principle

Instructions. If $kn+1$ objects ($k \geq 1$) are distributed among n boxes, then one of the boxes will contain at least $k+1$ objects.

1. Given a set of $n+1$ positive integers, none of which exceeds $2n$, show that at least one member of the set must divide another member of the set. [This could also be done by induction, but try pigeonhole.]

2. Consider any five points P_1, P_2, P_3, P_4, P_5 in the interior of a square of side length 1. Prove that for some pair P_i, P_j, $i \neq j$, the distance between P_i and P_j is less than $\sqrt{2}/2$.

3. Suppose that each square of a 3×7 checkerboard is colored either black or white. Prove that in any such coloring, the board must contain a rectangle (formed by horizontal and vertical lines of the board), such as the one outlined in the figure, whose distinct corner squares are all the same color.



4. Given any set of ten natural numbers between 1 and 99 inclusive, prove that there are two disjoint nonempty subsets of the set with equal sums of their elements.

5. Let x be any real number. Prove that among the numbers

$$x, 2x, \ldots, (n-1)x$$

there is one that differs from an integer by at most $1/n$.

2002 A2 Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

1995 B1 For a partition π of $\{1,2,3,4,5,6,7,8,9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

1990 A3 Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area greater than or equal to $5/2$.