

# Putnam Seminar — Problems 6

## INDUCTION

**Instructions.** Base case + induction step = good proof.

1. Prove the binomial theorem: For each integer  $n > 1$ ,

$$(a + b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}.$$

2. Prove that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

3. Prove that  $2!4! \cdots (2n)! \geq ((n+1)!)^n$ .

4. Let  $0 < a_1 < a_2 < \cdots < a_n$ , and let  $e_i = \pm 1$ . Prove that  $\sum_{i=1}^n e_i a_i$  assumes at least  $\binom{n+1}{2}$  distinct values as the  $e_i$  range over the  $2^n$  possible combinations of signs.

5. Prove that  $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$  is an integer for all natural numbers  $n$ .

6. Let  $a, b, p_1, p_2, \dots, p_n$  be real numbers with  $a \neq b$ . Define  $f(x) = (p_1 - x)(p_2 - x) \cdots (p_n - x)$ . Show that

$$\det \begin{bmatrix} p_1 & a & a & a & \cdots & a & a \\ b & p_2 & a & a & \cdots & a & a \\ b & b & p_3 & a & \cdots & a & a \\ b & b & b & p_4 & \cdots & a & a \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & b & \cdots & p_{n-1} & a \\ b & b & b & b & \cdots & b & p_n \end{bmatrix} = \frac{bf(a) - af(b)}{b - a}.$$

2003 B2 Let  $n$  be a positive integer. Starting with the sequence  $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$ , form a new sequence of  $n-1$  entries  $\frac{3}{4}, \frac{5}{12}, \dots, \frac{2n-1}{2n(n-1)}$  by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of  $n-2$  entries, and continue until the final sequence produced consists of a single number  $x_n$ . Show that  $x_n < 2/n$ .

2003 B3 Show that for each positive integer  $n$ ,

$$n! = \prod_{i=1}^n \text{lcm}\{1, 2, \dots, \lfloor n/i \rfloor\}.$$

(Here lcm denotes the least common multiple, and  $\lfloor x \rfloor$  denotes the greatest integer  $\leq x$ .)