Putnam Seminar — Problems 5

Exploit Symmetry

Instructions. The presence of symmetry in a problem usually provides a means for reducing the amount of work in arriving at a solution. Find algebraic or geometric symmetry in these problems to simplify them. Exploit!

1. Determine all values of x that satisfy

$$\tan x = \tan(x + 10^\circ) \tan(x + 20^\circ) \tan(x + 30^\circ).$$

   Hint: trig identities

$$\begin{aligned} \sin A \cos B &= \tfrac{1}{2}\left[\sin(A-B) + \sin(A+B)\right], \\ \sin A \sin B &= \tfrac{1}{2}\left[\cos(A-B) - \cos(A+B)\right], \\ \cos A \cos B &= \tfrac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]. \end{aligned}$$

2. Evaluate

$$\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$

3. Let P be a point on the graph of $y = f(x)$, where f is a third-degree polynomial; let the tangent at P intersect the curve again at Q; and let A be the area of the region bounded by the curve and the segment PQ. Similarly, let B be the area of the region bounded by the curve and the tangent line at Q. What is the relationship between A and B?

4. Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments AK, BK, BL, CL, CM, DM, DN, AN are the twelve vertices of a regular dodecagon.

5. Minimize $x_1^2 + x_2^2 + \ldots + x_n^2$, subject to the condition that $0 < x_i < 1$, and $x_1 + x_2 + \ldots + x_n = 1$.

1989 A2 Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx$, where a and b are positive.