Putnam Seminar — Problems 4

Work backward

Instructions. Try to solve these problems by assuming the conclusion and then deduce something that is known to be true. Reverse the steps to obtain a legit proof.

1. Let α be a fixed real number, $0 < \alpha < \pi$, and let
$$F(\theta) = \frac{\sin\theta + \sin(\theta+\alpha)}{\cos\theta - \cos(\theta+\alpha)}, \qquad 0 \le \theta \le \pi - \alpha.$$
Show that F is constant.

2. If a, b, c denote the lengths of the sides of a triangle, show that
$$3(ab+bc+ca) \le (a+b+c)^2 \le 4(ab+bc+ca).$$

3. Let AOB be a diameter of a circle with center O; BM is tangent to the circle at B; CF is tangent to the circle at E and meets BM at C; the chord AE when extended meets BM at D. Prove that $BC = CD$.

4. In a round-robin tournament with n players $P_1, P_2, \ldots, P_n$, where $n > 1$, each player plays one game with each of the other players and the rules are such that no ties can occur. Let W_r and L_r be the number of games won and lost, respectively, by player P_r. Show that
$$\sum_{r=1}^{n} W_r^2 = \sum_{r=1}^{n} L_r^2.$$

5. Find all positive integers n such that
$$3^n + 4^n + \cdots + (n+2)^n < (n+3)^n.$$

1998 A3 Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that
$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \ge 0.$$