Putnam Seminar — Problems 3

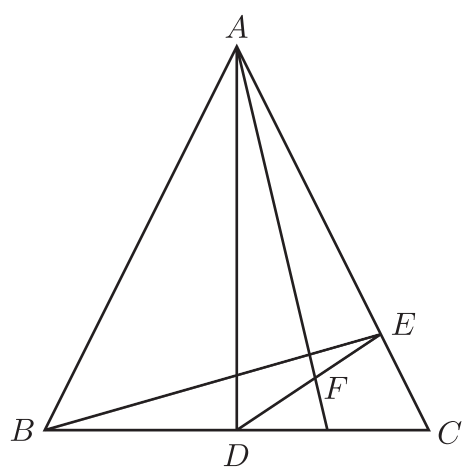
Choose effective notation

Instructions. Try to solve these problems by translating the problem into symbolic terms. Choose your notation to simplify the problem if possible. (u-substitution is a good example of this from calculus.)

1. One morning it started snowing at a heavy and constant rate. A snowplow started out at 8:00 A.M. At 9:00 A.M. it had gone 2 miles. By 10:00 A.M. it had gone 3 miles. Assuming that the snowplow removes a constant volume of snow per hour, determine how long (in hours) before 8:00 it started snowing.

2. If n is a positive integer such that $2n+1$ is a perfect square, show that $n+1$ is the sum of two successive perfect squares.

3. In triangle ABC, $AB = AC$, D is the midpoint of BC, E is the foot of the perpendicular drawn from D to AC, and F is the midpoint of DE. (See figure.) Prove that AF is perpendicular to BE.



4. Let $-1 < a_0 < 1$ and define recursively

$$a_n = \left(\frac{1 + a_{n-1}}{2}\right)^{1/2}, \qquad n > 0.$$

Let $A_n = 4^n(1 - a_n)$. What is $\lim_{n\to\infty} A_n$?

5. Prove that there exist infinitely many integers n such that n, $n+1$, $n+2$ are each the sum of two squares of integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.

1993 A2 Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1 \text{ for } n = 1, 2, 3 \dots.$$

Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.