Putnam Seminar — Problems 2

Draw a Figure

Instructions. Try to solve these problems by drawing a figure, diagram, graph or some other visual representation of the problem.

1. A chord of constant length slides around in a semicircle. The midpoint of the chord and the projections of its ends upon the base form the vertices of a triangle. Prove that the triangle is isosceles and its angles never change.

2. If a and b are positive integers with no common factor, show that

$$\left\lfloor \frac{a}{b} \right\rfloor + \left\lfloor \frac{2a}{b} \right\rfloor + \left\lfloor \frac{3a}{b} \right\rfloor + \cdots + \left\lfloor \frac{(b-1)a}{b} \right\rfloor = \frac{(a-1)(b-1)}{2},$$

   where $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.

3. Mr. and Mrs. Adams recently attended a party with three other couples. Various handshakes took place. No one shook hands with his/her partner, no one shook hands with the same person twice, and of course, no one shook his/her own hand. After all the handshaking was finished, Mr. Adams asked each person, including his wife, how many hands he or she had shaken. To his surprise, each gave a different answer. How many hands did Mr. Adams shake?

4. Two poles, with heights a and b, are a distance d apart (along level ground). A wire stretches from the top of each pole to some point P on the ground between them. Where should P be located to minimize the total length of the wire?

5. Let a and b be given positive real numbers with $a < b$. If two points are chosen randomly, uniformly on a line segment of length b, what is the probability that the distance between them is at least a?

2000 A-3 The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.