Putnam Seminar — Problems 1

Find a Pattern

Instructions. Try to solve these problems by searching for patterns. If you already know induction then prove these formally. This is a common technique: figure out what the answer probably is by considering small values of n and finding a pattern, then prove by induction or another counting argument.

1. How many subsets can be formed from a set of n objects?

2. A *composition* of the positive integer n is an ordered list of positive integers that sum to n. For example, the compositions of 3 are (3), (2,1), (1,2), (1,1,1). How many compositions of n are there?

3. Let $S_{n,0}$, $S_{n,1}$, $S_{n,2}$ denote the sum of every third element in the nth row of Pascal's triangle, beginning on the left with the first element, the second element, and the third element respectively. What is the value of $S_{100,1}$?

4. Let $x_1, x_2, x_3, \ldots$ be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}, \qquad n = 3, 4, 5, \ldots.$$

   Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

5. Find positive numbers n and $a_1, a_2, \ldots, a_n$ such that $a_1 + \ldots + a_n = 1000$ and the product $a_1 a_2 \ldots a_n$ is as large as possible.

6. Beginning with 2 and 7, the sequence $2, 7, 1, 4, 7, 4, 2, 8, \ldots$ is constructed by multiplying successive pairs of its members and adjoining the result as the next one or two members of the sequence, depending on whether the product is a one-or two-digit number. Prove that the digit 6 appears an infinite number of times in the sequence.

2007 B3 Let $x_0 = 1$ and for $n \geq 0$, let $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5} \rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}. ($\lfloor a \rfloor$ means the largest integer $\leq a$.)