

**The 69th William Lowell Putnam Mathematical Competition**  
**Saturday, December 6, 2008**

A1 Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function such that  $f(x, y) + f(y, z) + f(z, x) = 0$  for all real numbers  $x, y,$  and  $z$ . Prove that there exists a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(x, y) = g(x) - g(y)$  for all real numbers  $x$  and  $y$ .

A2 Alan and Barbara play a game in which they take turns filling entries of an initially empty  $2008 \times 2008$  array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

A3 Start with a finite sequence  $a_1, a_2, \dots, a_n$  of positive integers. If possible, choose two indices  $j < k$  such that  $a_j$  does not divide  $a_k$ , and replace  $a_j$  and  $a_k$  by  $\gcd(a_j, a_k)$  and  $\text{lcm}(a_j, a_k)$ , respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note:  $\gcd$  means greatest common divisor and  $\text{lcm}$  means least common multiple.)

A4 Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ xf(\ln x) & \text{if } x > e. \end{cases}$$

Does  $\sum_{n=1}^{\infty} \frac{1}{f(n)}$  converge?

A5 Let  $n \geq 3$  be an integer. Let  $f(x)$  and  $g(x)$  be polynomials with real coefficients such that the points  $(f(1), g(1)), (f(2), g(2)), \dots, (f(n), g(n))$  in  $\mathbb{R}^2$  are the vertices of a regular  $n$ -gon in counterclockwise order. Prove that at least one of  $f(x)$  and  $g(x)$  has degree greater than or equal to  $n - 1$ .

A6 Prove that there exists a constant  $c > 0$  such that in every nontrivial finite group  $G$  there exists a sequence of length at most  $c \ln |G|$  with the property that each element of  $G$  equals the product of some subsequence. (The elements of  $G$  in the sequence are not required to be distinct. A *subsequence* of a sequence is obtained by selecting some of the terms, not necessarily consecutive, without reordering them; for example, 4, 4, 2 is a subsequence of 2, 4, 6, 4, 2, but 2, 2, 4 is not.)

B1 What is the maximum number of rational points that can lie on a circle in  $\mathbb{R}^2$  whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

B2 Let  $F_0(x) = \ln x$ . For  $n \geq 0$  and  $x > 0$ , let  $F_{n+1}(x) = \int_0^x F_n(t) dt$ . Evaluate

$$\lim_{n \rightarrow \infty} \frac{n! F_n(1)}{\ln n}.$$

B3 What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

B4 Let  $p$  be a prime number. Let  $h(x)$  be a polynomial with integer coefficients such that  $h(0), h(1), \dots, h(p^2 - 1)$  are distinct modulo  $p^2$ . Show that  $h(0), h(1), \dots, h(p^3 - 1)$  are distinct modulo  $p^3$ .

B5 Find all continuously differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that for every rational number  $q$ , the number  $f(q)$  is rational and has the same denominator as  $q$ . (The denominator of a rational number  $q$  is the unique positive integer  $b$  such that  $q = a/b$  for some integer  $a$  with  $\gcd(a, b) = 1$ .) (Note:  $\gcd$  means greatest common divisor.)

B6 Let  $n$  and  $k$  be positive integers. Say that a permutation  $\sigma$  of  $\{1, 2, \dots, n\}$  is *k-limited* if  $|\sigma(i) - i| \leq k$  for all  $i$ . Prove that the number of  $k$ -limited permutations of  $\{1, 2, \dots, n\}$  is odd if and only if  $n \equiv 0$  or  $1 \pmod{2k + 1}$ .