Untangle: Knots in Combinatorial Game Theory

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Abstract. We introduce a new combinatorial game that involves some basic principles of knot theory. Two players begin with a projection of the unknot and take turns making Reidemeister moves until the projection is untangled. The last player to move is the winner. We analyze some families of games and raise questions, including possible connections with some open problems in knot theory.

1. Knots and Reidemeister Moves

A knotted piece of string with loose ends can always, in principle, be untied. But if we glue the ends together, we may not be able to untie the resulting closed loop. To study this phenomenon mathe-

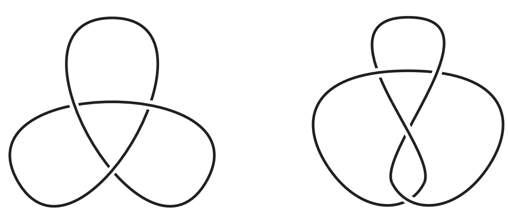

Figure 1. Mathematical knots.

matically, we can define a knot to be a smooth, simple closed curve in 3-dimensional space. Two knots are considered equivalent if one can be smoothly deformed into the other without cutting the knot or allowing it to cross through itself.

A knot is called *trivial* if it is equivalent to a circle; a trivial knot may also be called *the unknot*. We usually draw *projections* of

FIGURE 2. Three projections of the unknot.

knots, that is, diagrams in the plane in which one arc of the knot may appear to pass under or over another arc. A portion of the diagram where an underpass/overpass occurs is called a *crossing* of the projection. In this paper we will primarily be concerned with projections of the unknot, but a central problem in knot theory is to determine whether two given diagrams are projections of the same knot. For a good introductory treatment of knot theory, the reader is encouraged to consult [1], [6], or [7].

A projection can be deformed in simple ways without changing the knot. A deformation that takes place entirely in the plane and does not affect any crossings (as in the first two diagrams of Figure 2) is often called a *planar isotopy*. Three other simple deformations, known

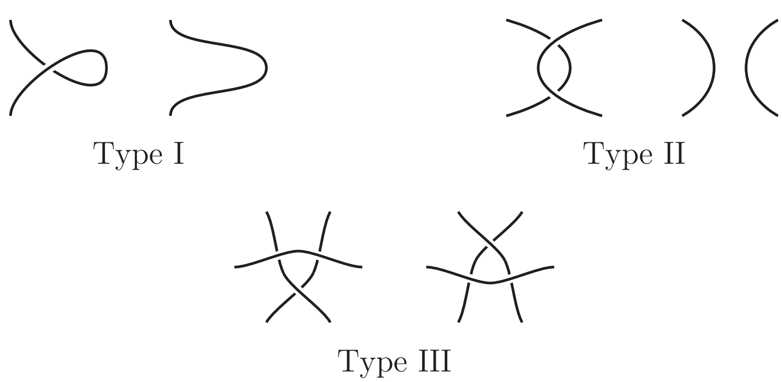

FIGURE 3. Reidemeister moves.

as *Reidemeister moves*, are illustrated in Figure 3. These diagrams are meant to represent parts of a larger knot projection. It should be clear that performing any of these moves: twisting or untwisting a loop (Type I), separating overlapping arcs or overlapping separated arcs (Type II), or moving an arc past a crossing (Type III), does not change the knot. Conversely, Reidemeister [8], and independently, Alexander and Briggs [3] proved that any two projections of a knot can be transformed from one to the other using only these moves, together with planar isotopies. In particular, any projection of the unknot can be transformed into a circle with a finite sequence of Reidemeister moves and planar isotopies. See Figure 4.

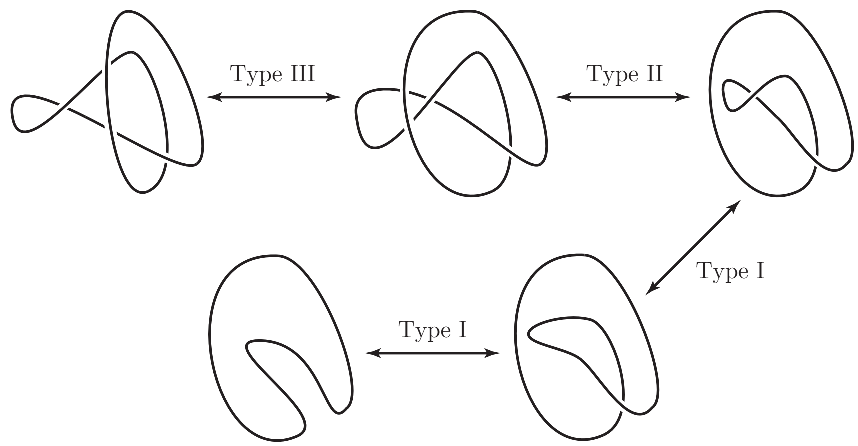

FIGURE 4. Transforming the unknot.

Note that it may be necessary to increase the number of crossings in a projection using Type II moves in order to untangle an unknot. Figure 5 shows such an example. We will call a Type I or Type II

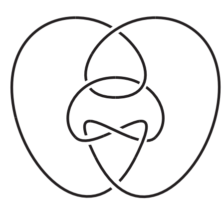

FIGURE 5. An unknot requiring *increasing* moves.

move *increasing* if it increases the number of crossings in the projection, otherwise we will call it *decreasing*. Notice that there are no decreasing Type I or Type II moves we can make in the example.

Also notice that there are no Type III moves at all. So to untangle this projection using Reidemeister moves we must increase the number of crossings in the projection before we decrease it.

Here are two interesting open questions concerning the unknot and Reidemeister moves.

Question 1. *Is there a positive integer k such that every n-crossing projection of the unknot can be untangled by a sequence of Reidemeister moves without passing through a projection with more than $n + k$ crossings?*

Question 2. *Is there a polynomial function $f(n)$ such that every n-crossing projection of the unknot can be untangled by a sequence of Reidemeister moves without passing through a projection with more than $f(n)$ crossings?*

2. Unknot Projections as Combinatorial Games

We now consider the process of untangling a given projection of the unknot from a perspective of combinatorial game theory. We describe a 2-player game in which each player takes turns reducing the number of crossings in a given unknot projection until no crossings remain, i.e., until the projection is planar isotopy equivalent to the circle. The game is called Untangle. The rules are as follows:

(1) A position consists of a projection of the unknot.
(2) Players take turns changing the projection using sequences of Reidemeister moves, subject to the restriction that the sequence of Reidemeister moves must be a minimal reducing sequence. That is, the sequence of Reidemeister moves must reduce the number of crossings, and if the sequence consists of m Reidemeister moves, there cannot be a sequence of fewer than m Reidemeister moves that would reduce the number of crossings.
(3) The game ends when there are no crossings remaining in the projection. The winner is the last player to move, that is, the player that untangles the projection.

The rules of Untangle guarantee that the game will end in a finite number of turns. The game is *impartial*, i.e., both players have the same available moves from every position. (Good references for the general theory of Combinatorial Games are [4] and [5].) Note that Rule 2 forces a turn to consist of a single Reidemeister move if a reducing Type I or Type II move is available, but if both such reducing moves are available, the player may make either move.

An example is illustrated in Figure 6. Notice that there are two

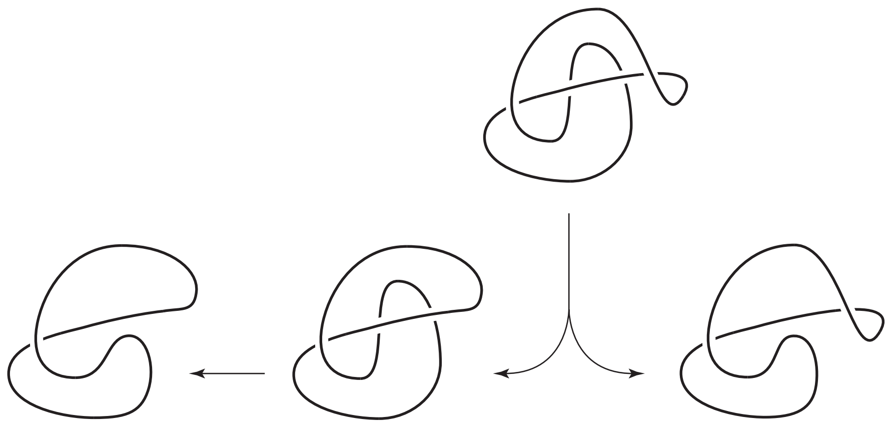

FIGURE 6. A game of UNTANGLE.

legal moves in the initial projection. If Player 1 begins with the available Type I move, she will force Player 2 to make a Type II move. Player 1 can then win with a Type I move. On the other hand, if Player 1 had begun with the available Type II move, Player 2 could then have won with a Type II move. We will always assume that players will play perfectly, i.e., will always follow a winning strategy if there is one. As in the standard terminology, we call a position in a game an *N-position* if perfect play will result in the next player winning from that position. The position is a *P-position* if perfect play will result in the previous player winning from that position. Given a position G, the *options* of G are the positions that can be reached from G in one legal move. Note that a position is an N-position if and only if at least one of its options is a P-position. Similarly, a position is a P-position if and only if all of its options are N-positions. Thus in the illustrated example, the starting projection is an N-position.

The *Grundy-value* (see [2], for example) $\mathcal{G}$ of a position G is calculated recursively as the smallest nonnegative integer that is not a Grundy value of any of the positions that are obtainable in one move from G, i.e., the minimum excluded value, or *mex* of the values of the options of G.

$$\mathcal{G}(G) = \text{mex}\,\{\mathcal{G}(H) \mid H \text{ is an option of } G\}.$$

A projection that is planar isotopy equivalent to the circle is a position with no options and so has Grundy-value 0. Notice that $\mathcal{G}(G) = 0$ if and only if G is a P-position. In the example in Figure 6, the position on the bottom right has three options. One is the unknot ($\mathcal{G} = 0$), and the others are one-crossing projections ($\mathcal{G} = 1$, since all subsequent positions are the unknot). So its Grundy-value is 2. The bottom center is a P-position, hence has Grundy-value 0. Thus the initial position has $\mathcal{G} = \text{mex}\{0, 2\} = 1$.

3. Analysis

The last move of the game must be a single reducing Type 1 or Type II move. (The situation in Figure 5 cannot occur with fewer than 7 crossings.) Thus we begin our analysis of the endgame by calculating Grundy-values for every projection with 2 or fewer crossings. Up to planar isotopy, there are four different positions with exactly one crossing. (Note that the two projections in Figure 7 are different.) Each of these has Grundy-value 1 since the reducing Type I

Figure 7. Two of the four 1-crossing projections.

move (the only legal move) wins the game. Of the ten projections with 2 crossings, one has Grundy-value 2; the rest are P-positions.

General analysis of this game appears difficult, and we have yet to determine which player has a winning strategy from a general position. We have done some analysis on the family of twist projections, which are constructed from a circle by a series of Type I moves (twisting left or twisting right) on a single arc. Figure 8 shows an

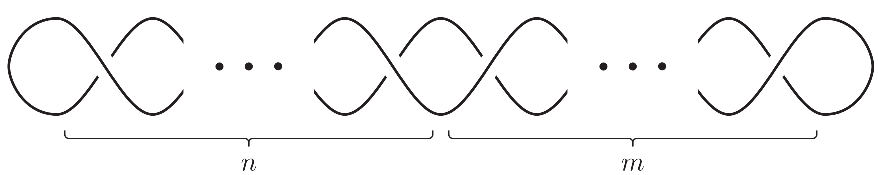

Figure 8. The $L^n R^m$ twist projection.

example with n left twists followed by m right twists. These $L^n R^m$

twist projections have Grundy-values

$$\mathcal{G}(L^n R^m) = 2\,(\min(n,m) \bmod 2) + ((n+m) \bmod 2).$$

Also, if the number of consecutive twists of each type in a position is even, for example the position $L^2R^6L^4$, then it is a P-position. We can find many more results like this through computer experimentation and induction. Nevertheless, it may very well be that, even in this special case, a general formula for Grundy-values or P-positions will prove elusive. Perhaps this game (like CHOMP) is a fruitful playground for "mathematical engineers" as described by Zeilberger in [9].

4. Questions

Question 3. *Are there any restrictions on the Grundy values of positions with n crossings? Perhaps a restriction on $\mathcal{G}$ as $n \to \infty$?*

Question 4. *Can we characterize the general twist projections*

$$L^{n_1}R^{m_1}L^{n_2}R^{m_2}\ldots L^{n_s}R^{m_s}$$

that are P-positions?

Question 5. *What modifications of the rules of* UNTANGLE *yield interesting variations? One suggestion is to make each turn consist of a single Reidemeister move. Decreasing moves must be made when possible, but moves that force a repeated position are illegal.*

Question 6. *Can information about* UNTANGLE *(or its variations) be used to gain insight into questions 1 or 2?*

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