UNORIENTED LINKS AND THE JONES POLYNOMIAL

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Abstract. The Jones polynomial is an invariant of oriented links with $n \geq 1$ components. When $n = 1$, the choice of orientation does not affect the polynomial, but for $n > 1$, changing orientations of some (but not all) components can change the polynomial. Here we define a version of the Jones polynomial that is an invariant of *unoriented* links, i.e., changing orientation of any sublink does not affect the polynomial. This invariant shares some, but not all of the properties of the Jones polynomial.

The construction of this invariant also reveals new information about the original Jones polynomial. Specifically, we show that the Jones polynomial of a knot is never the product of a nontrivial monomial with another Jones polynomial.

1. Introduction

Jones' original construction [4] of the polynomial $V_L = V_L(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ was through the skein relation

$$t^{-1}V_{L_-} - tV_{L_+} = (t^{1/2} - t^{-1/2})V_{L_0},$$

where L_+, L_- and L_0 are three oriented links that are identical except inside a ball that contains respectively, a positive crossing, a negative crossing, and two uncrossed strands. It is easy to see that when L is a knot (i.e., a link of one component), the polynomial $V_L(t)$ is unchanged by reversing the orientation on L, since crossing signs are preserved by such a change in orientation.

For links of more than one component, however, the Jones polynomial may change depending on the choice of orientation for each component. The Hopf link is the simplest example. The oriented Hopf link with linking number $+1$ has Jones polynomial $-t^{1/2} - t^{5/2}$, but reversing the orientation of one component gives us $-t^{-5/2} - t^{-1/2}$. A complete list of oriented links up to nine crossings, together with their polynomials can be found in [1].

Based on the skein-relation definition, it is a surprising result that a change in orientation of some components of L simply multiplies V_L by a power of t. Let $L = M \cup N$ be an oriented link with components $M_1, \ldots, M_r$,

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$N_1, \ldots, N_s$, and write $L_N = M \cup -N$ for the link formed by reversing the orientations on $N_1, \ldots N_s$. Morton [6] proved that

$$V_L(t) = t^{3\lambda} V_{L_N}(t),$$

where λ is the linking number of M with N, defined as

$$\lambda = \mathrm{lk}(M, N) = \sum_{i,j} \mathrm{lk}(M_i, N_j).$$

A much simpler proof using Kauffman's bracket polynomial construction of the Jones polynomial appears below.

Recall [5] the bracket polynomial $\langle L \rangle \in \mathbb{Z}[A, A^{-1}]$ is defined recursively:

$$\begin{aligned} \langle \times \rangle &= A \, \langle \asymp \rangle + A^{-1} \, \langle \rangle\langle \rangle \\ \langle \bigcirc L \rangle &= (-A^2 - A^{-2}) \, \langle L \rangle \\ \langle \bigcirc \rangle &= 1. \end{aligned}$$

The bracket polynomial is invariant under Reidemeister moves R2 and R3, but not under move R1. Define $X_L(A) = (-A^3)^{-w} \langle L \rangle$, where $w = w(L)$ is the writhe (sum of all crossing signs) of L, to obtain a link invariant. Under the change of variables $A = t^{-1/4}$, we have $X_L(A) = V_L(t)$. We will often write $d = -A^2 - A^{-2}$, thus $\langle \bigcirc L \rangle = d \, \langle L \rangle$.

Now it is clear that changing the orientations of some components of L multiplies the Jones polynomial by a power of t, since only the writhe (but not the bracket polynomial) is affected by such a change. Using the notation above, if $L = M \cup N$, then the crossing signs that change to produce L_N are the ones that involve some crossing of component M_i with component N_j. Since the linking number of M with N involves precisely the same crossings, we have

$$\begin{aligned} w(L_N) &= w(L) - 2 \sum (\text{crossing signs of } M_i \text{ with } N_j) \\ &= w(L) - 4 \cdot \mathrm{lk}(M, N). \end{aligned}$$

Thus

$$\begin{aligned} V_L(t) = X_L(A) &= (-A^3)^{-w(L)} \langle L \rangle \\ &= (-A^3)^{-4 \cdot \mathrm{lk}(M,N) - w(L_N)} \langle L \rangle \\ &= (-A)^{-12 \cdot \mathrm{lk}(M,N)} (-A^3)^{-w(L_N)} \langle L \rangle \\ &= (A^4)^{-3\lambda} X_{L_N}(A) \\ &= t^{3\lambda} V_{L_N}(t), \end{aligned}$$

confirming Morton's result.

Given an unoriented link of n components, there may be up to 2^{n-1} associated Jones polynomials for the links obtained by choosing an orientation for each component. (Note: Not 2^n, since changing *all* orientations does not affect the Jones polynomial.) None of these is a natural choice to be the

Jones polynomial of the unoriented link since there is no preferred orientation. In the next section we define a version of the Jones polynomial that is an invariant of unoriented links.

2. The Jones Polynomial for Unoriented Links

We begin by defining the *self-writhe* of a link diagram.

Definition 1. For a link diagram L with components $K_1, \ldots, K_n$, we define the self-writhe of L, $\psi(L)$ to be the sum of the writhes of each component of L, ignoring the other components when computing each writhe. That is,

$$\psi(L) = \sum_{j=1}^{n} w(K_j).$$

Equivalently, the self-writhe can be defined as the sum of the signs of those crossings of L for which both the under and over strands are from the same component.

Reidemeister moves affect the self-writhe exactly as they do the writhe. Both are invariant under moves R2 and R3. This is because the two crossings

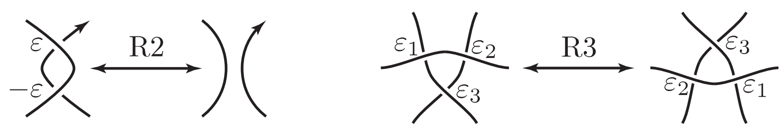


Figure 1. Crossing signs and Reidemeister moves.

involved in move R2 are of opposite sign regardless of orientation, and the crossing signs ε_i are unchanged by R3 moves regardless of orientations and components. See Figure 1. Under move R1, both the writhe and self-writhe change by ± 1, since move R1 always involves a single component of the link. See Figure 2.

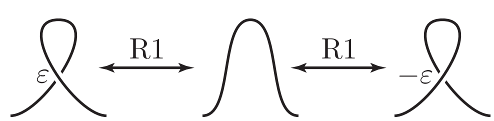


Figure 2. Crossing signs and move R1.

Unlike the writhe, however, the self-writhe of a link L is independent of the choice of orientations of the components of L. This is because changing the orientation of a component K of L does not affect the writhe of K, and hence does not affect $\psi(L)$.

Thus we can define $U_L(A) = (-A^3)^{-\psi} \langle L \rangle$. This modified Jones polynomial is an invariant for the same reason that $X_L(A)$ is: both $\langle L \rangle$ and

$\psi(L)$ are invariant under moves R2 and R3, and $\langle L \rangle$ changes by a factor of $(-A^3)^{\pm 1}$ with each R1 move.

But since $\psi(L)$ is unaffected by changing orientations of any components of L, the polynomial $U_L(A)$ is also unaffected by such changes. We can thus make the same change of variables $A = t^{-1/4}$ to obtain $W_L(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$, noting that $W_L(t) = U_L(A)$.

Definition 2. Let L be an unoriented link with self-writhe ψ. The Laurent polynomial $U_L(A) = (-A^3)^{-\psi} \langle L \rangle$ (or equivalently $W_L(t)$) for any choice of orientation of components of L is the unoriented Jones polynomial of L. We will refer to $U_L(A)$ as the U-polynomial of L.

3. Properties of the Unoriented Jones Polynomial

For knots we have $W_K(t) = V_K(t)$ since $w(K) = \psi(K)$. Thus we will examine the properties of the unoriented Jones polynomial for links of at least two components. Jones established [4] that if the link L has an odd number of components, then $V_L(t)$ is a Laurent polynomial over the integers; if the number of components of L is even then $V_L(t)$ is $\sqrt{t}$ times a Laurent polynomial. $W_L(t)$ does not share these properties. For example, if L is the Hopf link, then $W_L(t) = -t^{-1} - t$, since $\langle$  $\rangle = -A^4 - A^{-4}$, and $\psi($$) = 0$.

On the other hand, if L is link 5_1^2 (Figure 3), then $\psi(L) = w(L) = -1$,

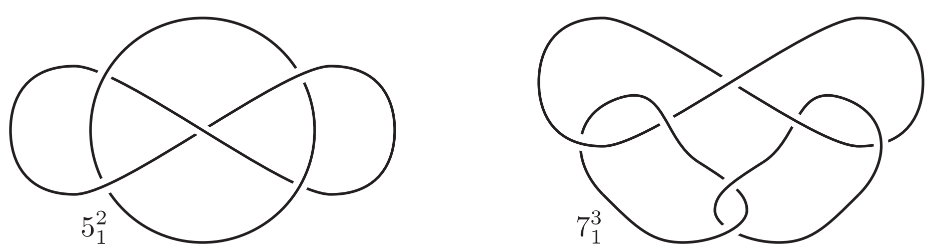


Figure 3. Links 5_1^2 and 7_1^3.

regardless of orientation. Therefore,

$$W_L(t) = V_L(t) = t^{-7/2} - 2t^{-5/2} + t^{-3/2} - 2t^{-1/2} + t^{1/2} - t^{3/2}.$$

There are two different oriented links corresponding to 7_1^3 (Figure 3), both of which have integral exponents for the original Jones polynomials, but the unoriented Jones polynomial is

$$t^{-5/2} - t^{-3/2} + 4t^{-1/2} - 3t^{1/2} + 4t^{3/2} - 3t^{5/2} + 3t^{7/2} - t^{9/2}.$$

For the remainder of this paper we use A as the indeterminate. This is simply to avoid fractional exponents.

Some properties of the Jones polynomial do carry over to $U_L(A)$. Let L^* denote the mirror image of L.

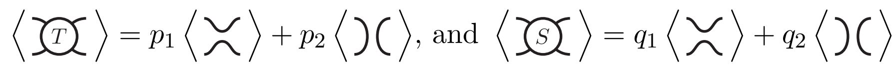
Proposition 3. $U_{L^*}(A) = U_L(A^{-1})$.

Proof. This follows immediately from definition 2, since $\psi(L) = -\psi(L^*)$. $\square$

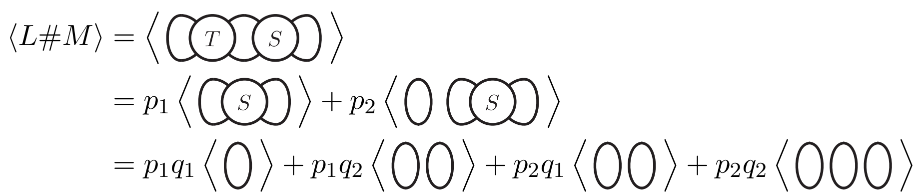
Theorem 4. *If L and M are links, then $U_{L\#M} = U_L U_M$.*

FIGURE 4. $L\#M$.

Proof. Observe that for diagrams L and M, the self-writhe of $L\#M$ is just $\psi(L) + \psi(M)$. Now take diagrams for L, M and $L\#M$ as in Figure 4. Let



where p_1, p_2, q_1 and q_2 are polynomials in A. Then we have $\langle L\rangle = p_1 + p_2 d$, and $\langle M\rangle = q_1 + q_2 d$. Moreover,



$$= p_1q_1 + p_1q_2d + p_2q_1d + p_2q_2d^2.$$

Thus,

$$\begin{aligned} U_{L\#M}(A) &= (-A^3)^{-\psi(L\#M)}\left(p_1q_1 + p_1q_2d + p_2q_1d + p_2q_2d^2\right) \\ &= (-A^3)^{-\psi(L)}(-A^3)^{-\psi(M)}(p_1 + p_2d)(q_1 + q_2d) \\ &= U_L(A)U_M(A). \end{aligned}$$ $\square$

When two links have the same number of components, their U-polynomials are related algebraically. Specifically, if L and L' are both n-component links, then $U(L) - U(L')$ is divisible by a certain fixed polynomial $C(A)$, independent of L, L' and n. Equivalently, we may say $U(L)$ and $U(L')$ are equal in the quotient ring $\mathbb{Z}[A, A^{-1}]/\langle C(A)\rangle$. For convenience, we will write $U(L) \equiv U(L') \pmod{C(A)}$.

Theorem 5. *Let L and L' be two links with the same number of components. Then $U_L(A) \equiv U_{L'}(A) \pmod{A^6 - 1}$.*

Proof. Suppose L and L' are two links that differ by a crossing change. We will show that $U_L(A) - U_{L'}(A)$ is divisible by $A^6 - 1$. Since any link can be transformed by crossing changes to any other link with the same number of components, the theorem follows.

Draw L and L' as the numerator closures of tangles that differ by a crossing as in Figure 5. Take the self-writhes of L and L' to be ψ and ψ'

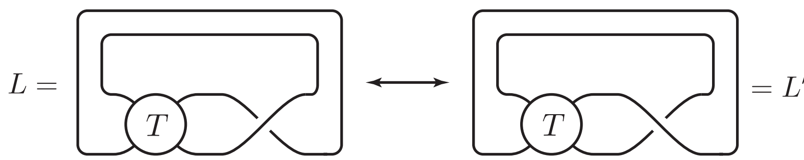
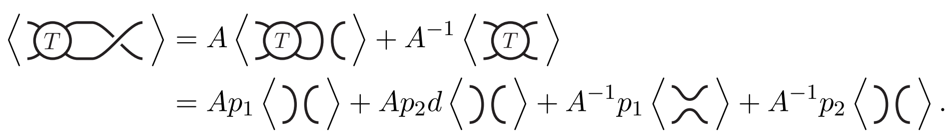


FIGURE 5. Two links that differ by a crossing change.

respectively. Therefore ψ' will equal ψ, $\psi+2$, or $\psi-2$, depending on the orientation of the strands in the crossing change, and whether they are from the same component. We compute $U_L(A) - U_{L'}(A)$. Write

where p_1 and p_2 are polynomials in A. Then



$$\begin{aligned}
\langle L\rangle &= Ap_1 + Ap_2d + A^{-1}p_1d + A^{-1}p_2 \\
&= p_1(A + A^{-1}d) + p_2(Ad + A^{-1}) \\
&= p_1(-A^{-3}) + p_2(-A^3). \\
U_L(A) &= (-A^3)^{-\psi}\left[p_1(-A^{-3}) + p_2(-A^3)\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle L'\rangle &= p_1(-A^3) + p_2(-A^{-3}), \\
U_{L'}(A) &= (-A^3)^{-\psi'}\left[p_1(-A^3) + p_2(-A^{-3})\right].
\end{aligned}$$

Since $\psi' \in \{\psi, \psi+2, \psi-2\}$, either

$$\begin{aligned}
U_L(A) - U_{L'}(A) &= (-A^3)^{-\psi}\left[p_1(-A^{-3} + A^3) - p_2(A^3 - A^{-3})\right] \\
&= (-1)^{-\psi}(A^3)^{-\psi-1}(p_1 - p_2)(A^6 - 1),
\end{aligned}$$

or

$$\begin{aligned}
U_L(A) - U_{L'}(A) &= (-A^3)^{-\psi}p_2(-A^3 + A^{-9}) \\
&= (-A^3)^{-\psi-3}p_2(A^6 + 1)(A^6 - 1),
\end{aligned}$$

or

$$\begin{aligned}
U_L(A) - U_{L'}(A) &= (-A^3)^{-\psi}p_1(-A^{-3} + A^9) \\
&= (-A^3)^{-\psi-1}p_1(A^6 + 1)(A^6 - 1).
\end{aligned}$$

□

Corollary 6. *Let L be a link with n components. Then $U_L(1) = (-2)^{n-1}$.*

Proof. Let $\bigcirc^n$ be the unlink of n components. Then

$$U_{\bigcirc^n}(A) = d^{n-1} = (-A^2 - A^{-2})^{n-1}.$$

Therefore by theorem 5, we can write $U_L(A) = (A^6 - 1)q(A) + (-A^2 - A^{-2})^{n-1}$, where q is some polynomial in A. Thus $U_L(1) = (-2)^{n-1}$. □

Theorem 5 establishes that $A^6 - 1$ divides the difference of any two U-polynomials of links with the same number of components. However, $A^6 - 1$ does not appear to be the highest-degree such polynomial. In all examples known to the authors, the difference is a multiple of $A^8 - A^6 - A^2 + 1$, which equals $(A^6 - 1)(A^2 - 1)$. We conjecture this is always the case.

Conjecture 7. Let L and L' be two links with the same number of components. Then $U_L(A) \equiv U_{L'}(A) \pmod{A^8 - A^6 - A^2 + 1}$.

We prove conjecture 7 for links of 3 or fewer components.

Theorem 8. *Let L and L' be two n-component links, where $n \leq 3$. Then $U_L(A) \equiv U_{L'}(A) \pmod{A^8 - A^6 - A^2 + 1}$.*

Proof. It is shown in [3] that when L is a knot (i.e., $n = 1$), then $X_L(A) - X_{L'}(A)$ (and hence $U_L(A) - U_{L'}(A)$) is always divisible by $A^{16} - A^{12} - A^4 + 1$, which equals $(A^8 - A^6 - A^2 + 1)(A^8 + A^6 + A^2 + 1)$.

For $n = 2$, we proceed as follows. It is proved in [7] that the link L can be transformed into the link L' by Δ-moves (Figure 6) if and only if L and

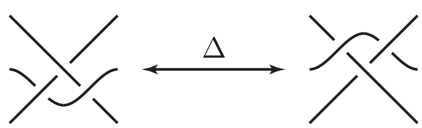


Figure 6. Δ-move.

L' have the same number of components and the pairwise linking numbers of the components of L equal those of L'. That is, if $L = K_1 \cup \cdots \cup K_n$ and $L' = K'_1 \cup \cdots \cup K'_{n'}$, then L can be transformed into L' by Δ-moves if and only if $n = n'$ and $\mathrm{lk}(K_i, K_j) = \mathrm{lk}(K'_i, K'_j)$ for $1 \leq i < j \leq n$. In this case we say L and L' are Δ-move equivalent. Thus every 2-component link is Δ-move equivalent to a link of the form in Figure 7, where $k \in \mathbb{Z}$ is the

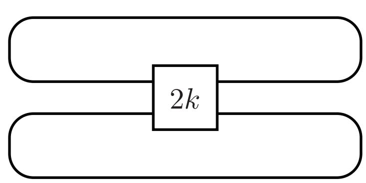
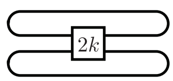
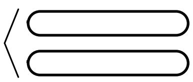
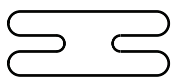


Figure 7. L_{2k}, a 2-component link with linking number k.

linking number.

It is shown in [3] that two links that differ by a sequence of Δ-moves have bracket polynomials that are congruent mod $A^8 - A^6 - A^2 + 1$ (in fact mod $A^{16} - A^{12} - A^4 + 1$). Since Δ-moves do not affect the self-writhe, the U-polynomials are also congruent mod $A^8 - A^6 - A^2 + 1$. Now, let L_{2k} be the link in Figure 7. We will show that $\langle L_{2k} \rangle - \langle \bigcirc\bigcirc \rangle$ is also a multiple of $A^8 - A^6 - A^2 + 1$. Thus every 2-component link has bracket polynomial congruent to $\langle \bigcirc\bigcirc \rangle \pmod{A^8 - A^6 - A^2 + 1}$. Since L_{2k} has self-writhe equal to 0, this will complete the proof.

We first compute $\langle L_{2k} \rangle$. We have

$\Big\langle$  $\Big\rangle = p_1 \Big\langle$  $\Big\rangle + p_2 \Big\langle$  $\Big\rangle$,

where $p_1 = A^{2k}$ and $p_2 = \sum_{m=1}^{2k} \binom{2k}{m} A^{2k-2m} d^{m-1}$. Now observe that

$$\begin{aligned} \sum_{m=0}^{2k} \binom{2k}{m} A^{2k-2m} d^m &= \sum_{m=0}^{2k} \binom{2k}{m} A^{2k-2m} (-A^2 - A^{-2})^m \\ &= \sum_{m=0}^{2k} \binom{2k}{m} A^{2k-m} (-A - A^{-3})^m \\ &= \left[(-A - A^{-3}) + A\right]^{2k} \end{aligned}$$

by the binomial theorem. The last expression simplifies to A^{-6k}. Therefore $p_2 = \frac{A^{-6k} - A^{2k}}{d}$, and

$$\begin{aligned} \langle L_{2k} \rangle &= A^{2k} d + \frac{A^{-6k} - A^{2k}}{d} \\ &= \frac{A^{2k}(A^4 + 2 + A^{-4}) + A^{-6k} - A^{2k}}{-A^2 - A^{-2}} \\ &= \frac{-A^{2k+6} - A^{2k+2} - A^{2k-2} - A^{-6k+2}}{A^4 + 1}. \end{aligned}$$

Thus,

$$\begin{aligned} \langle L_{2k} \rangle - \langle \bigcirc\bigcirc \rangle &= \frac{-A^{2k+6} - A^{2k+2} - A^{2k-2} - A^{-6k+2}}{A^4 + 1} + A^2 + A^{-2} \\ &= \frac{A^{8k+4} + A^{8k} + A^{8k-4} - A^{6k+4} - 2A^{6k} - A^{6k-4} + 1}{-A^{6k-2}(A^4 + 1)}. \qquad (1) \end{aligned}$$

Let $N(A)$ be the numerator of Equation (1). Since

$$A^8 - A^6 - A^2 + 1 = (A+1)^2 (A-1)^2 (A^2 + A + 1)(A^2 - A + 1),$$

we must show that $N(A)$ has these factors. (Actually, we only need to prove that 1 and -1 are double roots, since we have already established theorem 5. But it is not hard to show directly.) Rewrite $N(A)$ in the form

$$N(A) = (A^{8k+4} + A^{8k} + A^{8k-4}) - (A^{6k+4} + A^{6k} + A^{6k-4}) - (A^{6k} - 1).$$

Observe that

$$A^{8k+4} + A^{8k} + A^{8k-4} = A^{8k-4}(A^4 - A^2 + 1)(A^2 + A + 1)(A^2 - A + 1),$$
$$A^{6k+4} + A^{6k} + A^{6k-4} = A^{6k-4}(A^4 - A^2 + 1)(A^2 + A + 1)(A^2 - A + 1),$$

and

$$A^{6k} - 1 = (A^2 + A + 1)(A^2 - A + 1)\sum_{m=0}^{6k-6}(A^{m+2} - A^m).$$

It remains to show that $(A+1)^2$ and $(A-1)^2$ are factors of $N(A)$. It is straightforward to verify that 1 and -1 are both roots of $N(A)$ and of the derivative $N'(A)$, completing the proof for 2-component links.[1]

The proof for $n = 3$ is similar. Observe that every 3-component link is

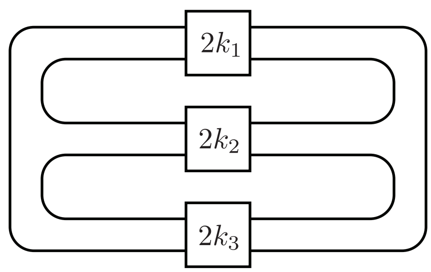


FIGURE 8. A 3-component link with linking numbers k_1, k_2, k_3.

Δ-move equivalent to a link of the form in Figure 8. Define

$$q(k) = \sum_{m=1}^{k} \binom{k}{m} A^{k-2m} d^{m-1},$$

so that

Then if L is the link in Figure 8, we have

$$\begin{aligned}\langle L\rangle = {} & A^{2k_1+2k_2+2k_3}d^2 \\ & + A^{2k_1+2k_2}q(2k_3)d + A^{2k_1+2k_3}q(2k_2)d + A^{2k_2+2k_3}q(2k_1)d \\ & + A^{2k_1}q(2k_2)q(2k_3) + A^{2k_2}q(2k_1)q(2k_3) + A^{2k_3}q(2k_1)q(2k_2) \\ & + q(2k_1)q(2k_2)q(2k_3)d,\end{aligned}$$

[1]Note that $N(A)$ must also be divisible by A^4+1, since bracket polynomials are Laurent polynomials. We can see this directly by writing

$$N(A) = (A^{8k} + A^{8k-4}) - (A^{6k+4} + A^{6k}) - (A^{6k} + A^{6k-4}) + (A^{8k+4} + 1).$$

The first three binomials are multiples of $A^4 + 1$, and

$$A^{8k+4} + 1 = (A^4 + 1)(A^{8k} - A^{8k-4} + A^{8k-8} - \cdots + 1).$$

and we must verify that $\langle L\rangle - \langle \bigcirc\bigcirc\bigcirc\rangle$ is divisible by $A^8 - A^6 - A^2 + 1$. The proof is tedious but elementary, and follows the same outline as for 2-component links. $\square$

Corollary 9. *For n-component links L, L' with $n \leq 3$, the U-polynomial of L can never be a nontrivial monomial times the U-polynomial of L'. I.e., if $U_{L'}(A) = rA^kU_L(A)$, then $r = 1$ and $k = 0$.*

Proof. Let $p(A) = A^8 - A^6 - A^2 + 1$, so that $U_L(A) - U_{L'}(A) = p(A)g(A)$ for some Laurent polynomial g. Now suppose $U_{L'}(A) = rA^kU_L(A)$. Then

$$U_L(A) - rA^kU_L(A) = p(A)g(A). \tag{2}$$

Setting $A = 1$, we obtain

$$(-2)^{n-1} - r(-2)^{n-1} = 0$$

from Corollary 6. Thus $r = 1$.

Differentiating Equation (2) with respect to A and setting $r = 1$, we obtain

$$(1 - A^k)U_L'(A) - kA^{k-1}U_L(A) = p'(A)g(A) + p(A)g'(A).$$

Again, setting $A = 1$ produces

$$kU_L(A) = 0.$$

Thus $k = 0$. $\square$

Corollary 9 does not hold for the original Jones polynomial. Example 10 below, shows a pair of 2-component links whose Jones polynomials do not satisfy the conclusion of the corollary. However, since the U-polynomial for a *knot* is identical to the original Jones polynomial, Corollary 9 does apply. Hence, the Jones polynomial of a knot cannot be the product of a nontrivial monomial with another Jones polynomial.

Example 10. In [2], examples are given of n-component links (for $n \geq 2$) that have the same Jones polynomial as $\bigcirc^n$. The link in Figure 9(a) is the first of an infinite family of such links. Those examples all have $w = \psi = 0$, therefore satisfy $U_L = U_{\bigcirc^n}$. Other examples are given in that paper of links whose Jones polynomial has the form t^kd^{n-1}, as in Figure 9(b). These links have $\psi = 0$, and as a result, $U(A) = d^{n-1}$.

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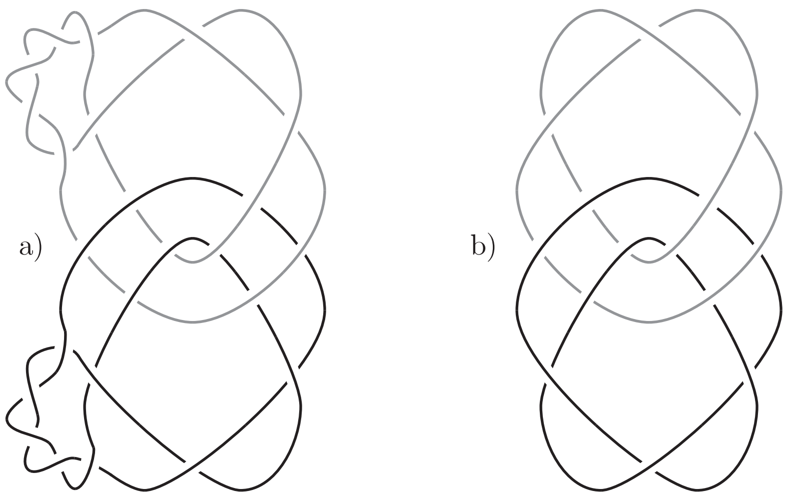

FIGURE 9. Links with $U_L = U_{\bigcirc^n}$.

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