Divisibility Tests, Old and New

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Tests for divisibility were once part of the standard curriculum. Checking the last digit of a number for divisibility by 2, 5 or 10 is still familiar for most, but fewer and fewer students are learning to check the last two digits for divisibility by 4, or the sum of the digits for divisibility by 3 or 9. And still fewer know why these tests work. Hardly anyone who isn't a working mathematician knows divisibility tests for 7 or 13.

But there's good mathematics behind these tests—mathematics worth learning and worth teaching—even in an age when our smartphones can check these things quickly. Modular arithmetic is central in many modern encryption techniques; working outside of base 10 is essential in computer science. Plus, the tests are fun, and interesting in their own right. And after nearly 2000 years of divisibility testing, we're still finding new ways to answer age-old questions.

1 Is n divisible by d?

The history of this question dates back at least to the Babylonian Talmud (Abodah Zarah 9b), where the reader is instructed that to determine whether $100a + b$ is divisible by 7, one need only check $2a + b$. The reason is that the two numbers differ by $98a$, which is a multiple of 7. But $2a + b$ is a smaller number, so easier to check for divisibility by 7. For example, to check whether 513 is a multiple of 7, we write

$$513 = (100 \times 5) + 13,$$

and compare with $(2 \times 5) + 13$, which equals 23. Since 23 isn't a multiple of 7, neither is 513. A thorough history of divisibility tests appears in [**?**].

In 1861, A. Zbikowski [**?**] published an elementary method for determining when any given integer is divisible by any other. A complete explanation of the technique can be found in [**?**], but the basic idea is described here.

Suppose we want to determine if a given number n is divisible by 21. We write $n = 10a + b$, where b is the last digit of n. Then we observe that

$$\begin{aligned} 10a + b &= 10a - 20b + 20b + b \\ &= 10(a - 2b) + 21b. \end{aligned}$$

So to determine whether n is a multiple of 21, we just need to find out whether $10(a-2b)$ is a multiple of 21. But since 21 and 10 have no common factors, we can just check whether $a - 2b$ is a multiple of 21. This is much smaller than n, and so it's easier to check. For example, to determine if 1967 is a multiple of 21, we write

$$1967 = (10 \times 196) + 7,$$

and compare with $196 - 14 = 182$. To check whether 182 is a multiple of 21 we can do the process again! We double the last digit and subtract it from the number formed by the remaining digits. We get $18 - 4 = 14$, which isn't a multiple of 21, so neither is 182. And so neither is 1967.

There are a two things worth noting about this technique. First, it doesn't only work as a divisibility test for 21; it also works as a divisibility test for any factor of 21. To test whether 1967 is divisible by 7, we only need to check whether 182 is divisible by 7. And for that we only need to know that 14 is a multiple of 7. So we conclude that 1967 is a multiple of 7. This is probably the most commonly-known divisibility test for 7 (other than just dividing the original number by 7).

Second, the technique can easily be modified to work for any number ending in 1. To test for divisibility by 31, write the given number n as $10a+b$ and then compare with $a-3b$ instead of $a-2b$. For example, is 2821 a multiple of 31? Multiply the last digit by 3 and subtract from the rest. We get $282 - 3 = 279$. Now repeat the process with 279. Since $27 - 27 = 0$, which is a multiple of 31, we conclude that 2821 is divisible by 31.

Now we have divisibility tests for all numbers that aren't multiples of 2 or 5. To test for divisibility by 17, we just find a multiple of 17 that ends in 1. Since $17 \times 3 = 51$, we take our number n, multiply the last digit by 5 and subtract from the number that remains. Is 2014 a multiple of 17? Just check $201 - 20 = 181$. Then check $18 - 5 = 13$. Since that's not a multiple of 17, neither is 181, nor 2014.

As for numbers that are multiples of 2 or 5, check the factors separately. To determine whether n is a multiple of 35, just check whether n is a multiple of both 5 and 7. (The test for 5 is easy.)

Here is an exercise: Zbikowski's test for divisibility by 19 would be challenging. That's because $19 \times 9 = 171$ is the smallest multiple of 19 that ends in a 1. We would have to take 17 times the last digit and subtract from the remaining number, which is hard if you don't know your multiples of 17. But there's an easier way! Zbikowski's test can be modified to give simple divisibility tests for numbers that end in 9. Can you see how?

2 But what's the remainder?

One drawback of Zbikowski's technique is that it typically doesn't give us the remainder when n is divided by d. In the divisibility by 17 example above, 2014 divided by 17 has a remainder of 8. That is, $2014 \equiv 8 \pmod{17}$. But when divided by 17, 181 has a remainder of 11, and 13 has a remainder of 13.

Note that the divisibility test for 7 from the Talmud *does* give us the correct remainder. When we divide 513 by 7 we get a remainder of 2, the same as when we divide 23 by 2. The reason is that in the Talmudic test we are simply subtracting a multiple of 7 (which keeps the same remainder), whereas in Zbikowski's test, we subtract a multiple of 17, but then divide the result by 10 (which doesn't).

The most familiar tests that do give us the correct remainder are the divisibility tests for 3 and 9. We just add up the digits. If the sum is a multiple of 9 then so is the original number. If not, then the remainder is the same as the remainder of the original. For example, to check whether 8,007,419,415 is a multiple of 9, we just take the sum $8+7+4+1+9+4+1+5 = 39$, which has a remainder of 3 when divided by 9. Thus $8007419415 \equiv 3 \pmod 9$. So it's not a multiple of 9, but it is a multiple of 3. Note that we could have used the test again on 39: $3+9 = 12$, and again on 12: $1+2 = 3$.

Let's see why this works. Choose a number n and look at its base-10 expansion. For example,

$$21568 = (2 \times 10^4) + (1 \times 10^3) + (5 \times 10^2) + (6 \times 10^1) + 8.$$

Note that for any integer $k \geq 1$,

$$10^k - 1 = \underbrace{999\ldots9}_{k} = 9 \times \underbrace{111\ldots1}_{k}.$$

So $10^k \equiv 1 \pmod 9$. Thus

$$\begin{aligned}
21568 &= \big((2\times 10^4)+(1\times 10^3)+(5\times 10^2)+(6\times 10^1)+8\big)\\
&\equiv \big((2\times 1)+(1\times 1)+(5\times 1)+(6\times 1)+8\big) \pmod 9\\
&\equiv (2+1+5+6+8) \pmod 9.
\end{aligned}$$

Of course, we can do this with any number: n is always congruent to the sum of its digits mod 9. And this gives us a divisibility test for all factors of 9 as well, since $a \equiv b \pmod{pq}$ implies $a \equiv b \pmod p$. Thus n is congruent to the sum of its digits mod 3.

Lagrange [**?**] made the observation that if we write the number n in base b, then the same congruence relation holds for $b-1$. For example, in base 8,

$$8^k - 1 = \underbrace{777\ldots7}_{k}{}_8 = 7 \times \underbrace{111\ldots1}_{k}{}_8.$$

Thus $8^k \equiv 1 \pmod 7$. So if we take the number 53103 in base 10, which is 147557_8 (in base 8), we can write

$$\begin{aligned}
147557_8 &= \big((1\times 8^5)+(4\times 8^4)+(7\times 8^3)+(5\times 8^2)+(5\times 8^1)+7\big)\\
&\equiv \big((1\times 1)+(4\times 1)+(7\times 1)+(5\times 1)+(5\times 1)+7\big) \pmod 7\\
&\equiv (1+4+7+5+5+7) \pmod 7\\
&\equiv 1 \pmod 7.
\end{aligned}$$

Which means that $53103 \equiv 1 \pmod 7$.

For many students, converting to a base b other than 10 is difficult, especially when $b > 10$. And in practice, it's no harder just to divide the original number by $b-1$, so this doesn't give us practical divisibility tests. But converting numbers to base 20, 30, 40, etc. can be done relatively easily (as we'll see below), so we can find good divisibility tests (that determine the remainder) for 19, 29, 39, etc., and any factors of those numbers (such as 13).

The key is the following algorithm for converting the number n to base b: Divide n by b and make note of the quotient q_1 and the remainder r_1. The last digit of n in base b will be r_1. (Do you see why?) Then divide q_1 by b to get the quotient q_2 and the remainder r_2 (which is the second to last digit of n in base b). Continue until the quotient is 0. Then $n = r_1 + br_2 + b^2r_3 + \cdots + b^{k-1}r_k$.

Example: Convert 2029_{10} to base 3. Begin by dividing 2029 by 3.

$$\begin{array}{r} 676 \text{ r } 1 \\ 3\,\overline{)\,2029} \end{array}$$

We get a quotient of 676 and a remainder of 1. Then divide 676 by 3 to get 225 with a remainder of 1. Continue dividing until the quotient is 0.

$$
\begin{array}{rl}
0 & \text{r } 2 \\
3\overline{)\,2} & \text{r } 2 \\
3\overline{)\,8} & \text{r } 1 \\
3\overline{)\,25} & \text{r } 0 \\
3\overline{)\,75} & \text{r } 0 \\
3\overline{)\,225} & \text{r } 1 \\
3\overline{)\,676} & \text{r } 1 \\
3\overline{)\,2029} &
\end{array}
$$

So

$$2029 = 1 + (1 \times 3) + (0 \times 3^2) + (0 \times 3^3) + (1 \times 3^4) + (2 \times 3^5) + (2 \times 3^6),$$

which means that $2029_{10} = 2210011_3$.

To convert to base 30 the algorithm looks like this:

$$
\begin{array}{rlr}
0 & \text{r} & 2 \\
30\overline{)\,2} & \text{r} & 7 \\
30\overline{)\,67} & \text{r} & 19 \\
30\overline{)\,2029} & &
\end{array}
$$

which tells us that $2029 = 19 + (7 \times 30) + (2 \times 30^2)$. But it's not easy for many students to do "short division" with 2-digit numbers. So here's a simplification that comes from the observation that $2029 \div 30 = 202.9 \div 3$. Start by dividing 202 by 3, getting 67 with a remainder of 1. The 1 becomes the first part of the base-30 "digit"; the 9 after the decimal point becomes the other part.

$$
\begin{array}{rl}
67\ \text{r}\ 1 & \longrightarrow 1\ 9 \\
3\overline{)\,202}.9 & \longrightarrow\ \ \uparrow
\end{array}
$$

Continue upward, dividing 6 by 3 (noting that $6.7 \div 3 = 67 \div 30$).

$$
\begin{array}{rl}
2\ \text{r}\ 0 & \longrightarrow 0\ 7 \\
3\overline{)\,6}.7 & \longrightarrow\ \ \uparrow
\end{array}
$$

Finally, $2 \div 30$ is 0 with remainder 2, or following the pattern,

$$
\begin{array}{rl}
0\ \text{r}\ 0 & \longrightarrow 0\ 2 \\
3\overline{)\,0}.2 & \longrightarrow\ \ \uparrow
\end{array}
$$

which gives us the same result as above. Adding the "digits" $2+7+19=28$ tells us that 2029 has remainder 28 when divided by 29. And in fact, $29 \times 70 = 2030$.

The divisibility test for 19 is even easier, since converting to base 20 requires only division by 2. Try it with 21506. You should get 17 as the remainder.

Let's do another example: we'll find the remainder when 21506 is divided by 13. Since $13 \times 3 = 39$, we'll convert to base 40 and add the digits mod 13.

$$
\begin{array}{rl}
0 \text{ r } 1 & \longrightarrow 1\,3 \\
4\overline{)\,1.3} \text{ r } 1 & \longrightarrow 1\,7 \\
4\overline{)\,5\,3.7} \text{ r } 2 & \longrightarrow 2\,6 \\
4\overline{)\,2\,1\,5\,0.6} &
\end{array}
$$

Since we're interested in the sum mod 13, we can ignore the 13 and 26 to get 4. Thus $21506 \div 13$ has a remainder of 4. We can also note, taking the sum mod 39, that $21506 \div 39$ has a remainder of 17.

The test for divisibility by 11 is similar to the one for 9, but instead of adding the digits, we alternately add and subtract them, starting with the ones digit. For example, to test 8,007,419,415 for divisibility by 11, we calculate

$$5-1+4-9+1-4+7-0+0-8=-5.$$

Since $-5 \equiv 6 \pmod{11}$, we conclude that $8007419415 \div 11$ has remainder 6. The reason this works is similar to the reason for divisibility by 9: powers of 10 are congruent to ± 1 mod 11. Specifically, even powers of 10 are congruent to 1 mod 11, whereas 10^1, 10^3, 10^5, etc. are congruent to -1 mod 11.

Like Lagrange's trick, this also works in any base: $b^k \equiv \pm 1 \pmod{(b+1)}$, the signs alternating for even and odd powers of b. Thus we have remainder-detecting divisibility tests for 21, 31, 41, etc., and any factors of those numbers. A quick check from the base-40 conversion above gives us $26-17+13=22$. So $21506 \div 41$ has a remainder of 22. And from the base-30 conversion, we get $19-7+2=14$. So $2029 \div 31$ has remainder 14.

Last example: What is the remainder when 2014 is divided by 17? Well, $17 \times 3 = 51$, and converting to base 50 we have $2014 = (40 \times 50) + 14$, so $14 - 40$ tells us the remainder mod 17. Since $-26 \equiv 8 \pmod{17}$, we have our answer. Lagrange would be proud.

References

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[4] Zbikowski, A. Note sur la divisibilité des nombres, Bull. Acad. Sci. St. Petersbourg 3 (1861) 151153.