DERIVATIVES OF JONES POLYNOMIALS DETECT DELTA MOVES IN VIRTUAL KNOTS

VICTORIA FURLOW, SANDY GANZELL, AND MADISON ROBINSON

Abstract. For classical knots, Δ-moves and crossing changes can both unknot any knot. But many virtual knots cannot be unknotted with these moves. Moreover, there are many virtual knots that can be unknotted by crossing changes but cannot be unknotted by Δ-moves. We show that the derivative of the Jones polynomial can detect whether a virtual knot can be unknotted by Δ-moves.

1. Introduction

A *local move* on a link diagram is a substitution, inside a prescribed ball, of one subdiagram for another, resulting in a diagram of a (possibly different) link. Local moves are considered "bidirectional" in the sense that if we permit the substitution of subdiagram X for subdiagram Y, then we also permit the substitution of Y for X. Reidemeister moves are local moves that preserve the link type, whereas a crossing change is local move that may change the link type.

We say that the local move M is an *unknotting* move if repeated uses of M (together with Reidemeister moves) can transform every knot diagram into a diagram of the unknot. It is a standard exercise to show that the crossing change is an unknotting move for classical knots. Satoh [17] has shown that crossing changes can also unknot any welded knot. However, crossing changes cannot unknot every virtual knot. The Kishino knot, shown in Figure 1, is a standard example. There

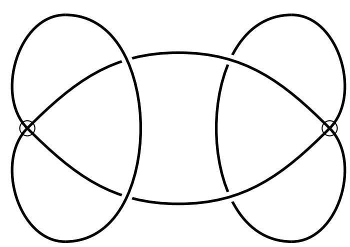

Figure 1. The Kishino knot.

are several ways (e.g., [3, 13, 2]) to show that the Kishino knot cannot be transformed into the unknot by crossing changes. Virtual knots that can be transformed to the unknot by crossing changes are sometimes called *homotopically trivial*.

The Δ-move (Figure 2) is also an unknotting move for classical knots [14] and for welded knots [17]. It is worth noting, however, that one cannot create an isolated

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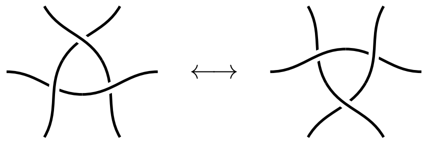


Figure 2. The Δ-move.

crossing change using Δ-moves. In particular, Δ-moves are not *unlinking* moves; they cannot transform every link into an unlink. It is easy to see that Δ-moves cannot change pairwise linking numbers among the components of a link. Hence for example, the Hopf link is not Δ-move equivalent to the two-component unlink. For convenience, when one can obtain knot K' from knot K using Δ-moves, we say K and K' are Δ-equivalent.

Of course, any virtual knot that cannot be unknotted by crossing changes (such as the Kishino knot) also cannot be unknotted by Δ-moves, since Δ-moves can be obtained by two crossing changes and a standard R3 Reidemeister move. However, there are many virtual knots that can be unknotted by crossing changes but cannot be unknotted with Δ-moves.

There is a simple invariant of virtual knots that can sometimes detect this phenomenon. The *odd writhe* [10] of a diagram is the sum of the signs of the odd crossings, i.e., those for which a strand from the crossing back to itself passes through classical crossings an odd number of times. The odd writhe is preserved by Reidemeister moves and thus is independent of the choice of diagram.

Proposition 1. *The odd writhe is unaffected by Δ-moves.*

Proof. The Gauss diagram prior to a Δ-move must include three arcs and three chords, where each arc contains an arrowhead and an arrowtail of distinct chords. The effect of the Δ-move is to switch the positions of the heads and tails on each of the three arcs as in Figure 3. Note that the parity of each crossing is unchanged.

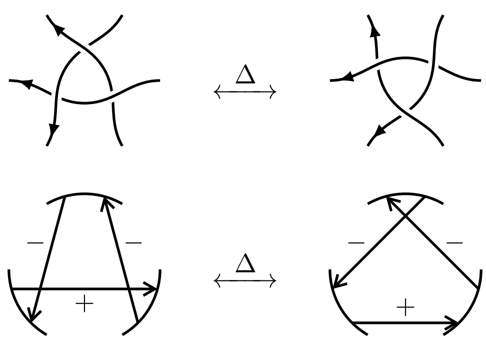


Figure 3. Effect of the Δ-move on the Gauss diagram.

That is, if the chord represents an odd crossing, it remains odd after the Δ-move, and conversely. Hence the odd writhe is unchanged by the Δ-move. $\square$

Example 2. Consider the virtual trefoil knot (Figure 4). Both crossings are odd,

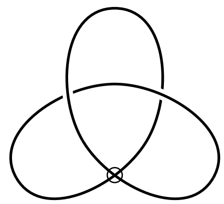


FIGURE 4. The virtual trefoil knot.

so the odd writhe is -2. Thus K cannot be unknotted by Δ-moves. But of course K can be unknotted by changing either classical crossing.

The remainder of the paper is organized as follows. We recall the basics of virtual knots, including the construction of the Jones polynomial in Section 2. In Section 3 we review a technique by the second author to relate Jones polynomials of classical knots that differ by Δ-moves. We then apply that technique to virtual knots to obtain the obstruction to virtual knots being unknotted by Δ-moves. In Section 4 we collect all the virtual knots of 4 crossings or fewer that can be unknotted by crossing changes but not by Δ-moves.

For virtual knots, we use the numbering and Gauss codes from Jeremy Green's Table of Virtual Knots [6].

2. Preliminaries

In [9], Kauffman introduced the theory of virtual knots. Like the classical theory, virtual knot theory has a useful diagrammatic approach. Virtual knot diagrams may be viewed as closed curves in the plane, with transverse crossings that have extra structure. In the classical theory, this extra structure is indicated by over and undercrossings. In the virtual theory, an additional, *virtual* crossing type is allowed, which is indicated in the diagrams with a small circle around the crossing. Equivalently, virtual knots may be interpreted as knots in thickened surfaces [1] or equivalence classes of Gauss data [7] as in Figure 5.

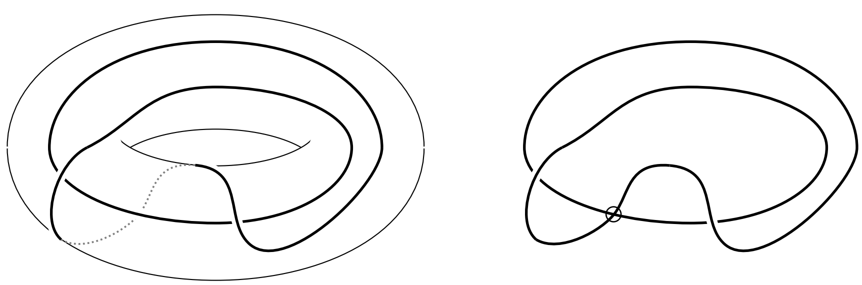


FIGURE 5. Virtual knot with Gauss code `O1+O2+U1+U2+`.

When viewed as diagrams with classical and virtual crossings, equivalence of virtual knots may be defined by means of a set of local moves (the extended Reidemeister moves) on their diagrams. See Figure 6. We then define a virtual knot to be an equivalence class of virtual diagrams modulo these moves (and planar isotopy). In [5] it is proved that two classical knots are equivalent under extended

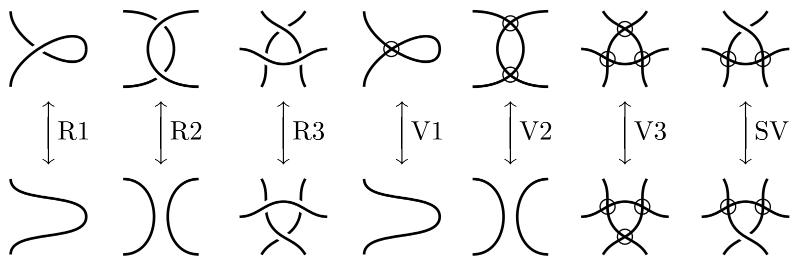


Figure 6. The extended Reidemeister moves.

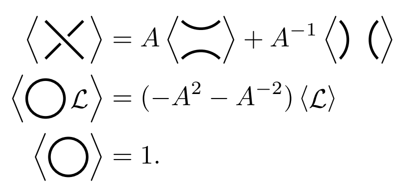
Reidemeister moves if and only if they are equivalent under classical Reidemeister moves. Thus virtual knot theory may be considered a generalization of the classical theory.

There are two additional Reidemeister-like moves, known as the *forbidden moves*, illustrated in Figure 7. Neither of these moves can be obtained as a sequence of

Figure 7. Forbidden moves.

extended Reidemeister moves. If we allow these moves, then any virtual knot can be transformed into any other virtual knot [11, 15], hence the designation of these moves as forbidden. If we allow one forbidden move but not the other, we obtain what are known as *welded knots* [16, 8].

The Jones polynomial can be defined for virtual knots through the Kauffman bracket exactly as it is for classical knots [9]. Specifically, for a diagram $\mathcal{D}$ of the virtual knot K, we define $\langle \mathcal{D} \rangle \in \mathbb{Z}[A, A^{-1}]$ by the recursion



The Jones polynomial of K is then defined by $f_K(A) = (-A^3)^{-w(\mathcal{D})}\langle \mathcal{D} \rangle$, where $w(\mathcal{D})$ is the writhe of $\mathcal{D}$. The (Laurent) polynomial $f_K(A)$ is independent of the choice of diagram, and we often use K to denote both the knot and the chosen diagram when no confusion results. We use the customary notation $d = -A^2 - A^{-2}$ so that $\langle \bigcirc\, K \rangle = d\langle K \rangle$.

3. Δ-moves and the Jones polynomial

The first theorem in this section is taken from [4]. It demonstrates a divisibility criterion for the Jones polynomial of classical knots established by the second author.

Theorem 3 ([4, Section 2])**.** *Let K_1 and K_2 be classical knots. Then $f_{K_1}(A) - f_{K_2}(A)$ is divisible by $A^{16} - A^{12} - A^4 + 1$.*

Proof. We show that any two knots that differ by a Δ-move have Jones polynomials whose difference is a multiple of $A^{16} - A^{12} - A^4 + 1$. Since Δ-moves are unknotting moves, every knot is Δ-equivalent to the unknot, and hence to every other knot. Thus there is a sequence of Δ-moves from K_1 to K_2. The theorem follows.

Let K and K' be two knots that differ by a Δ-move, as in Figure 8. We can

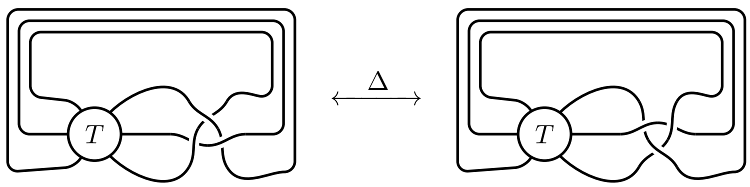
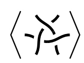
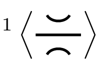
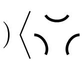
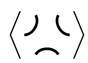
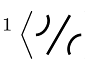
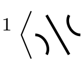
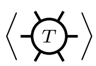
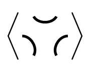
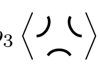
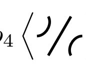
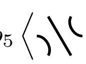


FIGURE 8. Two knots that differ by a Δ-move.

calculate the bracket:

 $= A^{-1}$  $+ (2A - A^5)$  $+ A^{-3}$  $+ A^{-1}$  $+ A^{-1}$  .

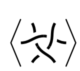
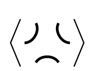
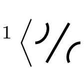
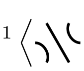
Now suppose

 $= p_1$  $+ p_2$  $+ p_3$  $+ p_4$  $+ p_5$  ,

where p_1, p_2, p_3, p_4 and p_5 are polynomials in A. Then

$$\begin{aligned}
\langle K\rangle &= p_1\big(A^{-1}d^2 + (2A - A^5)d + A^{-3}d + A^{-1} + A^{-1}\big)\\
&\quad + p_2\big(A^{-1}d + (2A - A^5)d^2 + A^{-3} + A^{-1}d + A^{-1}d\big)\\
&\quad + p_3\big(A^{-1}d + (2A - A^5) + A^{-3}d^2 + A^{-1}d + A^{-1}d\big)\\
&\quad + p_4\big(A^{-1} + (2A - A^5)d + A^{-3}d + A^{-1} + A^{-1}d^2\big)\\
&\quad + p_5\big(A^{-1} + (2A - A^5)d + A^{-3}d + A^{-1}d^2 + A^{-1}\big)\\
&= p_1(A^7 + A^{-1}) + p_2(-A^9) + p_3(-A^5 - A^{-3} + A^{-7})\\
&\quad + p_4(A^7 + A^{-1}) + p_5(A^7 + A^{-1}).
\end{aligned}$$

Similarly,

 $= A^{-1}$  $+ A^{-3}$  $+ (2A - A^5)$  $+ A^{-1}$  $+ A^{-1}$  ,

and

$$\begin{aligned}
\langle K'\rangle &= p_1(A^7 + A^{-1}) + p_2(-A^5 - A^{-3} + A^{-7}) + p_3(-A^9)\\
&\quad + p_4(A^7 + A^{-1}) + p_5(A^7 + A^{-1}).
\end{aligned}$$

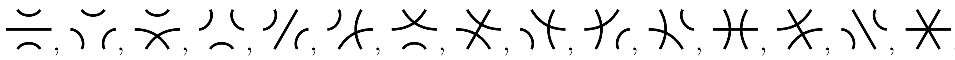
Notice that the writhe of K equals the writhe of K'. Letting the writhe equal w, we have

$$\begin{aligned}
f_K(A) - f_{K'}(A) &= (-A^3)^{-w}\big(\langle K\rangle - \langle K'\rangle\big)\\
&= (-A^3)^{-w}\big[p_2(-A^9 + A^5 + A^{-3} - A^{-7})\\
&\qquad - p_3(-A^9 + A^5 + A^{-3} - A^{-7})\big]
\end{aligned}$$

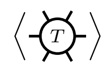
$$= (-A)^{-3w-7}(p_2 - p_3)(A^{16} - A^{12} - A^4 + 1),$$

as desired. □

Theorem 3 does not hold for virtual knots. There are two reasons for this. First, as mentioned earlier, Δ-moves are not unknotting moves for virtual knots. And second, referring again to Figure 8, the 3-tangle T may contain virtual crossings. Thus, for a similar statement about virtual knots, we must consider the 15 basic virtual 3-tangles (i.e., those without classical crossings or disjoint components), namely,



(All crossings in these 15 tangles should be taken as virtual crossings.) We may then write

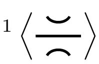
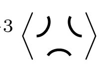
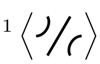
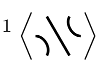
 $= \sum_{i=1}^{15} p_i \langle V_i \rangle$

where each p_i is a polynomial in A and the V_i are the basic virtual 3-tangles. Analogous to Theorem 3, we have the following.

Theorem 4. *Let K_1 and K_2 be virtual knots that differ by a sequence of Δ-moves. Then $f_{K_1}(A) - f_{K_2}(A)$ is divisible by $A^{12} + A^{10} - A^8 - 2A^6 - A^4 + A^2 + 1$.*

Proof. We show that any two virtual knots that differ by a single Δ-move have Jones polynomials whose difference is a multiple of $A^{12} + A^{10} - A^8 - 2A^6 - A^4 + A^2 + 1$. The theorem follows.

Let K and K' be two virtual knots that differ by a Δ-move, as in Figure 8. Since

 $= A^{-1}$  $+ (2A - A^5)$  $+ A^{-3}$  $+ A^{-1}$  $+ A^{-1}$ ,

we have

$$\begin{aligned}
\langle K \rangle = {} & p_1\big(A^{-1}d^2 + (2A - A^5)d + A^{-3}d + A^{-1} + A^{-1}\big) \\
& + p_2\big(A^{-1}d + (2A - A^5)d^2 + A^{-3} + A^{-1}d + A^{-1}d\big) \\
& + p_3\big(A^{-1}d + (2A - A^5)d + A^{-3} + A^{-1} + A^{-1}\big) \\
& + p_4\big(A^{-1}d + (2A - A^5) + A^{-3}d^2 + A^{-1}d + A^{-1}d\big) \\
& + p_5\big(A^{-1} + (2A - A^5)d + A^{-3}d + A^{-1} + A^{-1}d^2\big) \\
& + p_6\big(A^{-1} + (2A - A^5) + A^{-3}d + A^{-1} + A^{-1}d\big) \\
& + p_7\big(A^{-1}d + (2A - A^5) + A^{-3}d + A^{-1} + A^{-1}\big) \\
& + p_8\big(A^{-1} + (2A - A^5) + A^{-3} + A^{-1} + A^{-1}d\big) \\
& + p_9\big(A^{-1} + (2A - A^5)d + A^{-3} + A^{-1}d + A^{-1}\big) \\
& + p_{10}\big(A^{-1} + (2A - A^5)d + A^{-3} + A^{-1} + A^{-1}d\big) \\
& + p_{11}\big(A^{-1} + (2A - A^5) + A^{-3}d + A^{-1}d + A^{-1}\big) \\
& + p_{12}\big(A^{-1}d + (2A - A^5) + A^{-3} + A^{-1} + A^{-1}\big) \\
& + p_{13}\big(A^{-1} + (2A - A^5) + A^{-3} + A^{-1}d + A^{-1}\big) \\
& + p_{14}\big(A^{-1} + (2A - A^5)d + A^{-3}d + A^{-1}d^2 + A^{-1}\big) \\
& + p_{15}\big(A^{-1}d + (2A - A^5) + A^{-3} + A^{-1}d + A^{-1}d\big)
\end{aligned}$$

$$\begin{aligned}
&= p_1(A^7 + A^{-1}) + p_2(-A^9) + p_3(A^7 - A^3 - A) + p_4(-A^5 - A^{-3} + A^{-7}) \\
&\quad + p_5(A^7 + A^{-1}) + p_6(-A^5 + A + A^{-1} - A^{-3} - A^{-5}) \\
&\quad + p_7(-A^5 + A + A^{-1} - A^{-3} - A^{-5}) + p_8(-A^5 + A + 2A^{-1}) \\
&\quad + p_9(A^7 - A^3 - A) + p_{10}(A^7 - A^3 - A) \\
&\quad + p_{11}(-A^5 + A + A^{-1} - A^{-3} - A^{-5}) + p_{12}(-A^5 + A + 2A^{-1}) \\
&\quad + p_{13}(-A^5 + A + 2A^{-1}) + p_{14}(A^7 + A^{-1}) + p_{15}(-A^5 + A + 2A^{-3}).
\end{aligned}$$

Similarly, we calculate

$$\begin{aligned}
\langle K' \rangle &= p_1(A^7 + A^{-1}) + p_2(-A^5 - A^{-3} + A^{-7}) \\
&\quad + p_3(-A^5 + A + A^{-1} - A^{-3} - A^{-5}) + p_4(-A^9) + p_5(A^7 + A^{-1}) \\
&\quad + p_6(A^7 - A^3 - A) + p_7(A^7 - A^3 - A) + p_8(-A^5 + A + 2A^{-1}) \\
&\quad + p_9(-A^5 + A + A^{-1} - A^{-3} - A^{-5}) + p_{10}(-A^5 + A + A^{-1} - A^{-3} - A^{-5}) \\
&\quad + p_{11}(A^7 - A^3 - A) + p_{12}(-A^5 + A + 2A^{-1}) + p_{13}(-A^5 + A + 2A^{-1}) \\
&\quad + p_{14}(A^7 + A^{-1}) + p_{15}(-A^5 + A + 2A^{-3}).
\end{aligned}$$

Letting w be the writhe of K (which is also the writhe of K'), we have

$$\begin{aligned}
f_K(A) - f_{K'}(A) &= (-A^3)^{-w}\big(\langle K \rangle - \langle K' \rangle\big) \\
&= (-A^3)^{-w}\big(p_2(-A^9 + A^5 + A^{-3} - A^{-7}) \\
&\quad + p_3(A^7 + A^5 - A^3 - 2A - A^{-1} + A^{-3} + A^{-5}) \\
&\quad + p_4(-A^9 + A^5 + A^{-3} - A^{-7}) \\
&\quad + p_6(-A^7 - A^5 + A^3 + 2A + A^{-1} - A^{-3} - A^{-5}) \\
&\quad + p_7(-A^7 - A^5 + A^3 + 2A + A^{-1} - A^{-3} - A^{-5}) \\
&\quad + p_9(A^7 + A^5 - A^3 - 2A - A^{-1} + A^{-3} + A^{-5}) \\
&\quad + p_{10}(A^7 + A^5 - A^3 - 2A - A^{-1} + A^{-3} + A^{-5}) \\
&\quad + p_{11}(-A^7 - A^5 + A^3 + 2A + A^{-1} - A^{-3} - A^{-5})\big) \\
&= (-A^3)^{-w}\big(-A^{-7}p_2(A^{16} - A^{12} - A^4 + 1) \\
&\quad + A^{-5}p_3(A^{12} + A^{10} - A^8 - 2A^6 - A^4 + A^2 + 1) \\
&\quad - A^{-7}p_4(A^{16} - A^{12} - A^4 + 1) \\
&\quad - A^{-5}p_6(A^{12} + A^{10} - A^8 - 2A^6 - A^4 + A^2 + 1) \\
&\quad - A^{-5}p_7(A^{12} + A^{10} - A^8 - 2A^6 - A^4 + A^2 + 1) \\
&\quad + A^{-5}p_9(A^{12} + A^{10} - A^8 - 2A^6 - A^4 + A^2 + 1) \\
&\quad + A^{-5}p_{10}(A^{12} + A^{10} - A^8 - 2A^6 - A^4 + A^2 + 1) \\
&\quad + A^{-5}p_{11}(A^{12} + A^{10} - A^8 - 2A^6 - A^4 + A^2 + 1)\big) \\
&= (-A)^{-3w-7}\big(A^{12} + A^{10} - A^8 - 2A^6 - A^4 + A^2 + 1\big) \\
&\qquad \big((A^4 - A^2 + 1)p_2 - A^2p_3 + (A^4 - A^2 + 1)p_4 \\
&\qquad\qquad + A^2p_6 + A^2p_7 - A^2p_9 - A^2p_{10} - A^2p_{11}\big),
\end{aligned}$$

as claimed. □

Corollary 5. *If the virtual knot K is Δ-equivalent to the unknot then*

$$\frac{\mathrm{d}}{\mathrm{d}A} f_K(A)\Big|_{A=i} = 0.$$

Proof. Let us denote the polynomial $A^{12} + A^{10} - A^8 - 2A^6 - A^4 + A^2 + 1$ by $\mathfrak{I}(A)$. Since the unknot has Jones polynomial 1, it follows from Theorem 4 that if $f_K(A) - 1$ is not a multiple of $\mathfrak{I}(A)$ then K cannot be unknotted with Δ-moves.

It is shown in [4] that the difference of Jones polynomials for any two virtual knots is a multiple of $A^{10} - A^6 - A^4 + 1$. Hence for any virtual knot K, we may write $f_K(A) = (A^{10} - A^6 - A^4 + 1)q(A) + 1$ for some polynomial q. In particular, $f_K(i) = 1$. And since

$$\begin{aligned}\mathfrak{I}(A) &= (A^2+1)^2(A^2+A+1)(A^2-A+1)(A-1)^2(A+1)^2 \\ &= (A^2+1)(A^{10} - A^6 - A^4 + 1),\end{aligned}$$

we have that a necessary condition for K to be unknotable by Δ-moves is that $f_K(A) - 1$ has a double root at $A = i$. Since $f_K(i) - 1 = 0$, this is equivalent to $f'_K(i) = 0$. □

Example 6. Let K be knot 3.2 (Figure 9). Then $f_K(A) = A^8 - A^4 - A^2 + 1 + A^{-2}$,

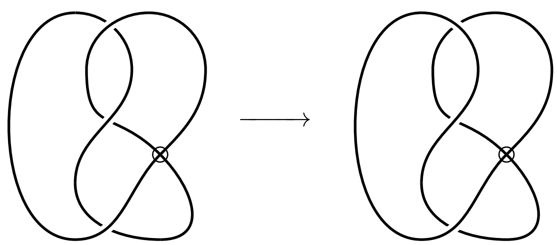


FIGURE 9. Virtual knot 3.2 is homotopically trivial.

so $f'_K(i) = -8i$. Thus K cannot be unknotted by Δ-moves. But it is easy to unknot K by changing the top classical crossing.

The situation in Example 6 is not unusual. As we see in Section 4, most of the homotopically trivial knots of four crossings or fewer cannot be unknotted with Δ-moves.

4. KNOTS WITH FOUR OR FEWER CROSSINGS

The converse of Corollary 5 is false. For example, Miyazawa's polynomial [13] shows that knot 3.1, whose Gauss code is `O1-O2-U1-O3+U2-U3+`, is homotopically nontrivial. Thus it cannot be Δ-equivalent to the unknot. But its Jones polynomial is trivial and hence satisfies $f'_K(i) = 0$.

However, if K is homotopically trivial, and $f'_K(i) = 0$, the result is inconclusive. As we see in the following two examples, knots 4.12 and 4.86 are both homotopically trivial, satisfy $f'_K(i) = 0$, and are Δ-equivalent to the unknot. But there are some homotopically trivial knots we are unable to determine whether they are Δ-equivalent to the unknot.

Example 7. Consider knot 4.12, pictured in Figure 10. We evaluate the derivative of the Jones polynomial:

$$f_K'(i) = -6A^5 - 4A^3 + 2A - 2A^{-3} + 4A^{-5} + 6A^{-7}\big|_{A=i} = 0.$$

As we see in the Gauss diagram, we can perform a Δ-move with crossings 1, 2, 4. Crossings 2 and 3 can then be eliminated with one R2-move, followed by a second

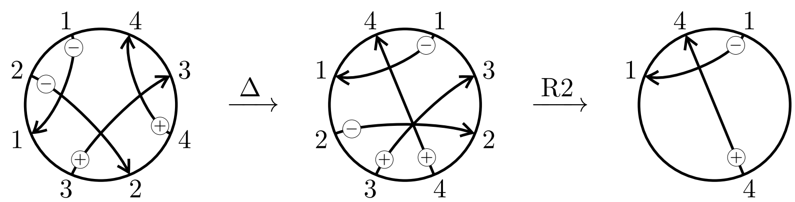


FIGURE 10. Unknotting 4.12 with a Δ-move.

R2-move to eliminate crossings 1 and 4.

Example 8. Knot 4.86 is similar. Its Jones polynomial is $A^8 - A^4 + 1 - A^{-4} + A^{-8}$, whose derivative vanishes at $A = i$. It can be unknotted with two Δ-moves, as seen in Figure 11. First we perform a Δ-move with crossings 1, 3, 4, followed by a second

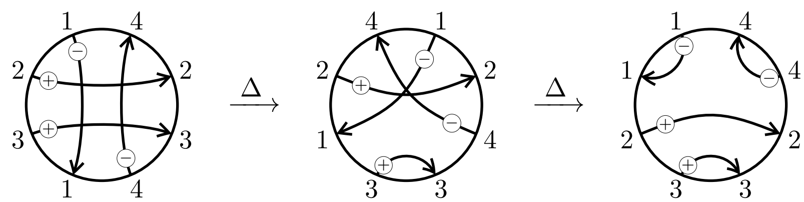


FIGURE 11. Knot 4.86 is Δ-equivalent to the unknot.

Δ-move with crossings 1, 2, 4. Four R1-moves then complete the transformation.

Of the 116 nontrivial virtual knots of four or fewer crossings, 42 can be unknotted by crossing changes, and 76 cannot (verifiable by Miyazawa's polynomial or Manturov's parity bracket [12]). We are unable to determine whether the remaining 5 knots, 4.85, 4.89, 4.90, 4.98 and 4.107, are homotopically trivial. All five of these satisfy $f_K'(i) = 0$.

Among the 42 knots that we know to be homotopically trivial, 24 satisfy the condition $f_K'(i) \neq 0$, and thus cannot be unknotted with Δ-moves. Of the remaining 18 knots, only 4 are definitely Δ-equivalent to the unknot: the classical trefoil and figure-8 knots (3.6 and 4.108), plus virtual knots 4.12 and 4.86 from Examples 7 and 8 above. The other 14 are unknown at this time. This information is collected in Table 1.

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knot	$f'_K(i)=0$	Δ-trivial	knot	$f'_K(i)=0$	Δ-trivial
2.1	no	no	4.65	yes	?
3.2	no	no	4.68	yes	?
3.5	yes	?	4.73	no	no
3.6	yes	yes	4.74	no	no
3.7	yes	?	4.75	yes	?
4.3	no	no	4.82	no	no
4.6	yes	?	4.84	no	no
4.12	yes	yes	4.86	yes	yes
4.25	no	no	4.91	no	no
4.27	no	no	4.92	no	no
4.36	yes	?	4.94	no	no
4.37	no	no	4.95	no	no
4.40	no	no	4.96	yes	?
4.41	yes	?	4.99	yes	?
4.43	no	no	4.100	no	no
4.44	no	no	4.101	no	no
4.46	yes	?	4.102	yes	?
4.53	no	no	4.104	no	no
4.54	no	no	4.105	yes	?
4.61	no	no	4.106	yes	?
4.64	no	no	4.108	yes	yes

TABLE 1. Homotopically trivial knots.

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