CHIRALITY VS. HOMFLY AND KAUFFMAN POLYNOMIALS

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Abstract. We exhibit an infinite family of knots that are detected chiral by the Kauffman polynomial but not by the HOMFLY polynomial.

1. Introduction

In [2], Kauffman introduced the polynomial that has since come to bear his name. In that paper it is noted that the Kauffman polynomial is very good at detecting chirality, in particular, it is better in this respect than the HOMFLY polynomial in many cases. In this paper we exhibit an infinite family of knots that lend support to Kauffman's statement. Namely, we show that each member of the family $\{P_k\}$ of $(2, 1-2k, 1+2k)$ pretzel knots $(k \geq 2)$ is detected chiral by the Kauffman polynomial but not by the HOMFLY polynomial. Whether there exists any knot detected chiral by HOMFLY but not by Kauffman is still open.

2. Computation of the Polynomials

2.1. **HOMFLY polynomial.** We use the construction of the HOMFLY polynomial defined by the axioms

$$
\begin{aligned}
\langle \bigcirc \rangle &= 1 \\
\langle K_+ \rangle - \langle K_- \rangle &= z \langle K_0 \rangle \\
\langle \quad \rangle &= a \langle \text{——} \rangle \\
\langle \quad \rangle &= a^{-1} \langle \text{——} \rangle ,
\end{aligned}
$$

where $\langle K \rangle$ is a regular isotopy invariant. The HOMFLY polynomial is then defined by $G(K) = a^{-w(K)} \langle K \rangle$, where $w(K)$ is the writhe of K. See [1] or [2] for details of this construction. Notice that $w(P_k) = 0$ for all $k \geq 2$, so $G(P_k) = \langle P_k \rangle$. For a knot K, $G(K) \in \mathbb{Z}\left[a^{\pm 2}, z\right]$, and if K^* is the mirror image of K then $G(K^*)$ is obtained from $G(K)$ by replacing a with $-a^{-1}$. For convenience of notation we define $G_K(a, z) = G(K)$ and let $\delta = \frac{a - a^{-1}}{z}$ so that $\langle \bigcirc K \rangle = \delta \langle K \rangle$. (Notice δ is unchanged by replacing a with $-a^{-1}$.)

In all of the equations below, the box represents the number of positive (right-handed) *half* twists of the strands.

We begin by calculating $\langle P_k \rangle$ recursively:

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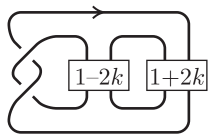
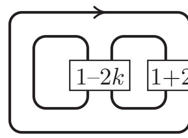
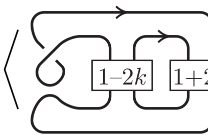
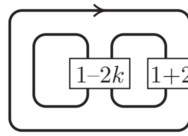
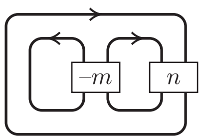
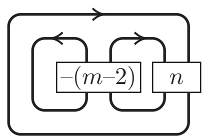
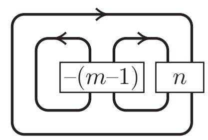
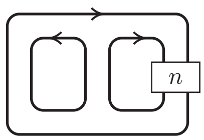
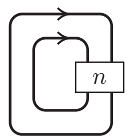
$\Big\langle$  $\Big\rangle$ $=$ $\Big\langle$  $\Big\rangle - z \Big\langle$  $\Big\rangle$

$=$ $\Big\langle$  $\Big\rangle - a^{-1}\delta z - z^2.$

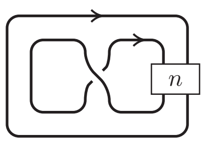
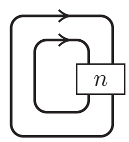
Now, for $m \geq 2$,

$\Big\langle$  $\Big\rangle = \Big\langle$  $\Big\rangle - z \Big\langle$  $\Big\rangle,$

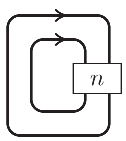
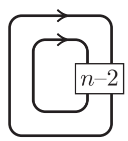
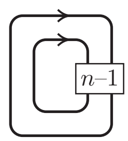
When $m = 0$ we have

$\Big\langle$  $\Big\rangle = \delta \Big\langle$  $\Big\rangle,$

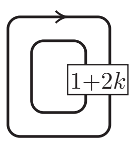
and when $m = 1$,

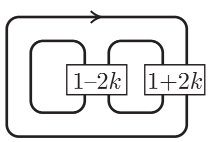
$\Big\langle$  $\Big\rangle = a^{-1} \Big\langle$  $\Big\rangle.$

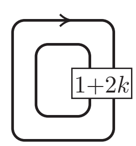
To complete the recursion, we calculate

$\Big\langle$  $\Big\rangle = \Big\langle$  $\Big\rangle + z \Big\langle$  $\Big\rangle$ for $n \geq 2$.

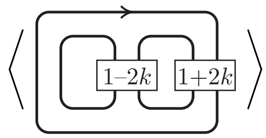
When $n = 0$ and $n = 1$ the bracket yields δ and a respectively. So by induction, we have

$\Big\langle$  $\Big\rangle$ $= a\sum_{i=0}^{k}\binom{k+i}{2i}z^{2i} + \delta\sum_{i=0}^{k-1}\binom{k+i}{2i+1}z^{2i+1}$, and

$\Big\langle$  $\Big\rangle$ $=$

$\left[a^{-1}\sum_{i=0}^{k-1}\binom{k+i-1}{2i}z^{2i} - \delta\sum_{i=0}^{k-2}\binom{k+i-1}{2i+1}z^{2i+1}\right]\Big\langle$  $\Big\rangle.$

Thus

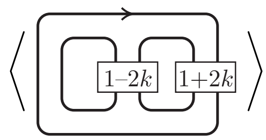


$$=$$

$$\left[a^{-1}\sum_{i=0}^{k-1}\binom{k+i-1}{2i}z^{2i}-\delta\sum_{i=0}^{k-2}\binom{k+i-1}{2i+1}z^{2i+1}\right]\left[a\sum_{i=0}^{k}\binom{k+i}{2i}z^{2i}+\delta\sum_{i=0}^{k-1}\binom{k+i}{2i+1}z^{2i+1}\right].$$

(It follows that $G_{P_k}(a,z)$ always has the form $c_{2,k}a^2+c_{0,k}+c_{-2,k}a^{-2}$, where each $c_{i,k}$ is a function in z.) We now show that $G_{P_k}(a,z)=G_{P_k}(-a^{-1},z)$.

Since δ is unchanged by the substitution of $-a^{-1}$ for a, we write



$$= f(\delta,z)+g(a,\delta,z)$$

where

$$\begin{aligned} g(a,\delta,z) &= a^{-1}\delta\left[\sum_{i=0}^{k-1}\binom{k+i-1}{2i}z^{2i}\right]\left[\sum_{i=0}^{k-1}\binom{k+i}{2i+1}z^{2i+1}\right] \\ &\quad -a\delta\left[\sum_{i=0}^{k}\binom{k+i}{2i}z^{2i}\right]\left[\sum_{i=0}^{k-2}\binom{k+i-1}{2i+1}z^{2i+1}\right]. \end{aligned}$$

It follows that the coefficients of $a^{-1}\delta z^{2s+1}$ and $-a\delta z^{2s+1}$ (in terms of k) are

$$\sum_{i=0}^{s}\binom{k+i-1}{2i}\binom{k-i+2}{2s+1-2i} \tag{2.1}$$

and

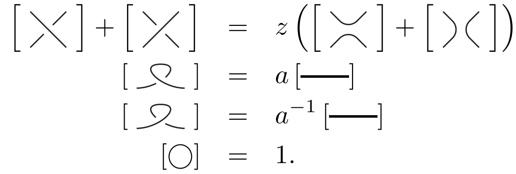
$$\sum_{i=0}^{s}\binom{k+i}{2i}\binom{k-i+1}{2s+1-2i} \tag{2.2}$$

respectively. When $s=1$ these are easily seen to be equal, and by induction we have that 2.1 and 2.2 are equal for $s\geq 1$. Plugging in $s=0$, however, we find that k and $k-1$ are the coefficients of $a^{-1}\delta z$ and $-a\delta z$ respectively. Thus

$$\begin{aligned} G_{P_k}(a,z) &= f(\delta,z)+g(a,\delta,z)-a^{-1}\delta z-z^2 \\ &= f(\delta,z)+g(-a^{-1},\delta,z)+a^{-1}\delta z+a\delta z-a^{-1}\delta z-z^2 \\ &= G_{P_k}(-a^{-1},z). \end{aligned}$$

Hence the HOMFLY polynomial does not detect that P_k is chiral.

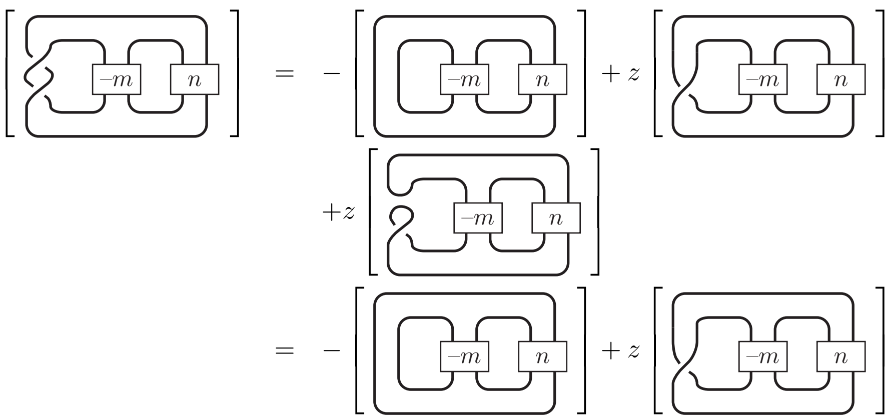
2.2. **Kauffman Polynomial.** Recall now that the Kauffman polynomial ([2]) is defined in terms of the regular isotopy invariant $[K]$ satisfying the axioms



The Kauffman polynomial is then defined by $F(K) = F_K(a,z) = a^{-w(K)}\,[K]$. For a knot K, $F(K) \in \mathbb{Z}\left[a^{\pm 1}, z\right]$, and $F_K(a,z) = F_{K^*}(a^{-1}, z)$. Again, each P_k has zero writhe, so $F(P_k) = [P_k]$. It is convenient to let $d = \frac{a+a^{-1}}{z} - 1$ so that $[\bigcirc K] = d\,[K]$.

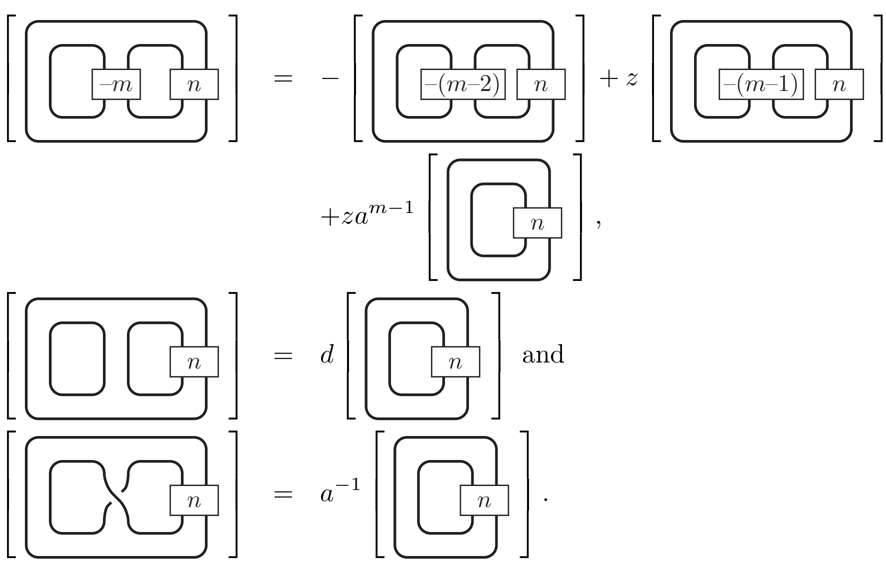
Here we do not endeavour to find an explicit expression for $F_{P_k}(a,z)$. Instead, we calculate the term of $F_{P_k}(a,z)$ with the highest power of z. This turns out to have the form cz^{4k} where c is a function for which $c(a) \neq c(a^{-1})$, verifying that the Kauffman polynomial does detect chirality for this family.

Again we proceed recursively. We have,

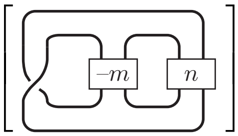


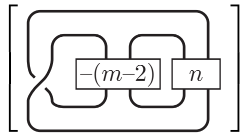
$$-a^{-1}zd + a^{-2}z^2 + z^2.$$

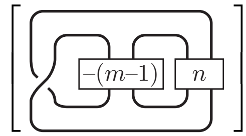
Now,

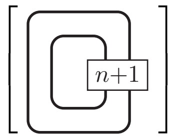


Similarly,



$=\ -$


$+\,z$


$+a^{m-1}z$


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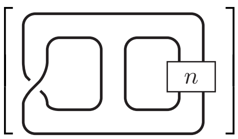
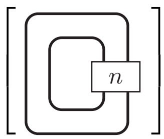


$=\ a$


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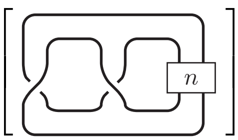
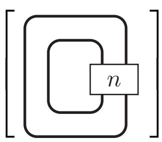
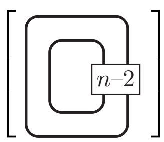
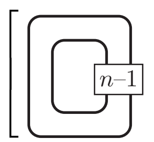


$=\ a^{-n}.$

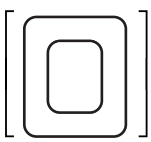
So to complete the recursion we compute

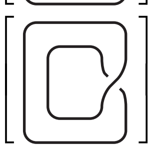


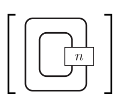
$=\ -$


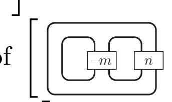
$+\,z$


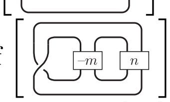
$+\,a^{1-n}z,$


$=\ d,$


$=\ a.$

Observe that for $n \geq 2$, the term of


with the highest power of z is $(a+a^{-1})z^{n-1}$. Thus for $m, n \geq 2$, the term of


with the highest power of z is $(a^2+2+a^{-2})z^{m+n-2}$, and the term of


with the highest power of z is $(a^2+1)z^{n+m-1}$. Hence for all k, the term of $[P_k]$ with the highest power of z is $(a^2+1)z^{4k}$, which implies that $F_{P_k}(a,z) \neq F_{P_k}(a^{-1},z)$.

References

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