RESTRICTIONS ON HOMFLYPT AND KAUFFMAN POLYNOMIALS ARISING FROM LOCAL MOVES

SANDY GANZELL, MERCEDES GONZALEZ, CHLOE' MARCUM, NINA RYALLS, AND MARIEL SANTOS

Abstract. We study the effects of certain local moves on Homflypt and Kauffman polynomials. We show that all Homflypt (or Kauffman) polynomials are equal in a certain nontrivial quotient of the Laurent polynomial ring. As a consequence, we discover some new properties of these invariants.

1. Introduction

The Jones polynomial has been widely studied since its introduction in [5]. Divisibility criteria for the Jones polynomial were first observed by Jones in [6], who proved that $1 - V_K$ is a multiple of $(1-t)(1-t^3)$ for any knot K. The first author observed in [3] that when two links L_1 and L_2 differ by specific local moves (e.g., crossing changes, Δ-moves, 4-moves, forbidden moves), we can find a fixed polynomial P such that $V_{L_1} - V_{L_2}$ is always a multiple of P. Additional results of this kind were studied in [11] (C_n-moves) and [1] (double-Δ-moves).

The present paper follows the ideas of [3] to establish divisibility criteria for the Homflypt and Kauffman polynomials. Specifically, we examine the effect of certain local moves on these invariants to show that the difference of Homflypt polynomials for n-component links is always a multiple of $a^4 - 2a^2 + 1 - a^2z^2$, and the difference of Kauffman polynomials for knots is always a multiple of $(a^2+1)(a^2+az+1)$.

In Section 2 we define our main terms and recall constructions of the Homflypt and Kauffman polynomials that are used throughout the paper. The primary theorem of Section 3 establishes the divisibility criterion described above for Homflypt polynomials. We then study consequences of that result, which reveal properties of the Homflypt polynomial and its derivatives, including a proof that a Homflypt polynomial can never be a nontrivial monomial. In Section 4 we prove the analogous theorems for the Kauffman polynomial.

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2. Preliminaries

A *local move* on a link diagram is a substitution, inside a prescribed ball, of one subdiagram for another, resulting in another link diagram. Local moves are considered "bidirectional" in the sense that if we permit the substitution of subdiagram X for subdiagram Y, then we also permit the substitution of Y for X. Reidemeister moves are local moves that preserve the link type, whereas a crossing change is a local move that may change the link type.

We say that the local move M is an *unknotting* move if repeated uses of M (together with Reidemeister moves) can transform every knot diagram into a diagram of the unknot. We say that M is an *unlinking* move if repeated uses of M (together with Reidemeister moves) can transform every n-component link diagram into a diagram of the n-component unlink. For example, the standard crossing change is an unlinking move, but the Δ-move (Figure 1) is

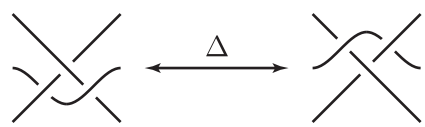


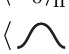
Figure 1. The Δ-move.

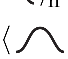
not, since it does not affect the pairwise linking numbers of its components. The Δ-move is, however, an unknotting move, as shown in [10]. Thus we say that every knot is Δ-equivalent to the unknot.

We assume the reader is familiar with Kauffman's bracket construction of the Jones Polynomial [8]. For a diagram L, we denote the bracket of L by $\langle L\rangle \in \mathbb{Z}[A, A^{-1}]$. Recall that the bracket is a *regular* isotopy invariant (i.e., invariant under Reidemeister moves II and III). The Jones polynomial is then given by $V_L = V_L(A) = (-A^3)^{-w}\langle L\rangle$, where w is the writhe of L. We use $A = t^{-1/4}$ for the indeterminate, and let $d = -A^2 - A^{-2}$, so that $\langle \bigcirc L\rangle = d\langle L\rangle$.

The Homflypt polynomial $G_L = G_L(a, z)$ is most commonly defined [2] by a skein relation in two variables: $aG_{L_+} - a^{-1}G_{L_-} = zG_{L_0}$, where G_L is an invariant normalized to equal 1 for the unknot. The links L_+, L_- and L_0 as usual are identical except at a single crossing site. However, we may equivalently define $G_L(a, z)$ by a bracket $\langle L\rangle_{\mathrm{h}}$ according to the rules:

$$\langle L_+\rangle_{\mathrm{h}} - \langle L_-\rangle_{\mathrm{h}} = z\langle L_0\rangle_{\mathrm{h}}$$

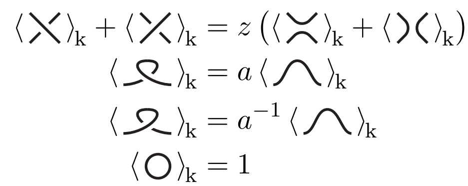
$\langle$  $\rangle_{\mathrm{h}} = a\,\langle$  $\rangle_{\mathrm{h}}$

$\langle$  $\rangle_{\mathrm{h}} = a^{-1}\,\langle$  $\rangle_{\mathrm{h}}$

$$\langle \bigcirc \rangle_{\mathrm{h}} = 1$$

where $\langle L\rangle_{\mathrm{h}}$ is a regular isotopy invariant. The Homflypt polynomial is then defined by $G_L = a^{-w}\langle L\rangle_{\mathrm{h}}$, where w is the writhe of L. See [7] for details of this construction. It is convenient to let $\delta = \frac{a-a^{-1}}{z}$ so that $\langle \bigcirc L\rangle_{\mathrm{h}} = \delta\langle L\rangle_{\mathrm{h}}$.

The bracket construction of the Homflypt polynomial is analogous to the standard construction of the Kauffman polynomial $F_L = F_L(a, z)$, which is defined by a bracket $\langle L\rangle_{\mathrm{k}}$ according to the rules:



where $\langle L\rangle_{\mathrm{k}}$ is also a regular isotopy invariant. The Kauffman polynomial is then defined by $F_L = a^{-w}\langle L\rangle_{\mathrm{k}}$, where w is again the writhe of L. We let $\lambda = \frac{a+a^{-1}}{z} - 1$ so that $\langle \bigcirc L\rangle_{\mathrm{k}} = \lambda\langle L\rangle_{\mathrm{k}}$.

As usual, the entire knot is not always drawn in a bracket calculation; two diagrams in a calculation should be presumed identical outside of the portion drawn.

3. Local Moves and Homflypt Polynomials

In this section we establish the main divisibility criterion for the Homflypt polynomial, namely that the Homflypt polynomials of any two n-component links ($n \geq 1$) are equal in a certain quotient of the ring $\mathbb{Z}[a, a^{-1}, z, z^{-1}]$. Recall [9] that for a knot K, we have $G_K \in \mathbb{Z}[a^2, a^{-2}, z^2]$, and more generally, the Homflypt polynomial of an n-component link is an element of $\mathbb{Z}[a, a^{-1}, z, z^{-1}]$, with powers of a and z being odd or even when n is even or odd respectively. The lowest power of z is $1 - n$.

Theorem 1. *Let L_1 and L_2 be n-component links. Then $G_{L_1} - G_{L_2}$ is a multiple of $a^4 - 2a^2 + 1 - a^2z^2$.*

Proof. We will show that when two diagrams differ by a crossing change, then the difference of their Homflypt polynomials is a multiple of $a^4 - 2a^2 + 1 - a^2z^2$. Since one can convert any link into any other link with the same number of components using a sequence of crossing changes, the result follows.

Any oriented link diagram with a crossing can be drawn as one of the diagrams in Figure 2. Label these L_1 and L_2 respectively, where L_1 has the

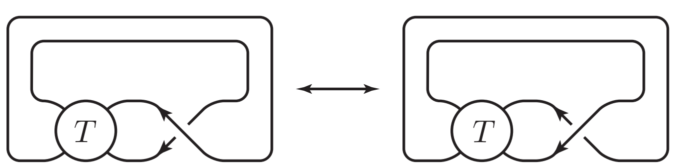
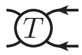


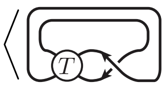
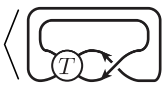
Figure 2. Two links that differ by a crossing change.

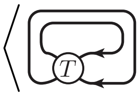
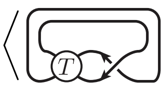
positive crossing. Let w be the writhe of L_1, so $w - 2$ is the writhe of L_2.

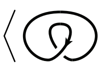
By repeated use of the definition of $G_L(a,z)$, we may write

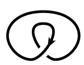
$\Big\langle$  $\Big\rangle_{\mathrm{h}} = p_0 \Big\langle$  $\Big\rangle_{\mathrm{h}} + p_1 \Big\langle$  $\Big\rangle_{\mathrm{h}}$,

where p_0 and p_1 are polynomials in a and z. Thus we have

$G_{L_1} - G_{L_2} = a^{-w}\Bigg(\Bigg\langle$  $\Bigg\rangle_{\mathrm{h}} - a^2 \Bigg\langle$  $\Bigg\rangle_{\mathrm{h}}\Bigg)$

$= a^{-w}\Bigg(z\Bigg\langle$  $\Bigg\rangle_{\mathrm{h}} + (1-a^2)\Bigg\langle$  $\Bigg\rangle_{\mathrm{h}}\Bigg)$

$= a^{-w}\Bigg(zp_0\Big\langle$  $\Big\rangle_{\mathrm{h}} + zp_1\Big\langle$  $\Big\rangle_{\mathrm{h}}$

$+ (1-a^2)p_0\Big\langle$  $\Big\rangle_{\mathrm{h}} + (1-a^2)p_1\Big\langle$  $\Big\rangle_{\mathrm{h}}\Bigg)$

$$
\begin{aligned}
&= a^{-w}\Big(zp_0\delta + zp_1 a^{-1} \\
&\qquad + (1-a^2)p_0 a^{-1} + (1-a^2)p_1\big((a-a^{-1})z^{-1} - a^{-1}z\big)\Big) \\
&= a^{-w}\big(p_0(a - a^{-1} + a^{-1} - a) \\
&\qquad + p_1(za^{-1} + 2az^{-1} - a^{-1}z^{-1} - a^3z^{-1} - a^{-1}z + az)\big) \\
&= -a^{-w-1}z^{-1}p_1\big(a^4 - 2a^2 + 1 - a^2z^2\big),
\end{aligned}
$$

as desired. □

The polynomial $a^4 - 2a^2 + 1 - a^2z^2$ appears throughout this paper. We denote it by $P_h(a,z)$:

$$P_h(a,z) = a^4 - 2a^2 + 1 - a^2z^2 = (a^2 - 1 - az)(a^2 - 1 + az). \tag{1}$$

Note that $P_h(a,z)$ is precisely the difference between the Homflypt polynomial of the unknot and that of the right-handed trefoil. Thus $P_h(a,z)$ is maximal in the sense that any polynomial that divides the difference of every pair of Homflypt polynomials will divide $P_h(a,z)$.

Corollary 2. *The Homflypt polynomial of a knot has the general form $P_h(a,z)g(a,z) + 1$. More generally, the Homflypt polynomial of a link with n components has the general form $P_h(a,z)g(a,z) + \delta^{n-1}$.*

Proof. This follows immediately from Theorem 1 since the Homflypt polynomial of the n-component unlink is δ^{n-1}. □

Corollary 3. *For any link L, if $z = \frac{a^2-1}{a}$, then $G_L(a,z) = 1$.*

Proof. From (1), we see that $P_h(a,z) = 0$ under the given substitution. Moreover, since $\delta = \frac{a-a^{-1}}{z}$, we have that $\delta = 1$. Thus the general form of Corollary 2 gives us the result. □

Theorem 1 gives us general information regarding Homflypt polynomials and their derivatives. If we consider Homflypt polynomials to be elements of $\big(\mathbb{Z}[a,a^{-1}]\big)[z,z^{-1}]$, then the leading coefficient (in this z-expansion) is a (Laurent) polynomial in the indeterminate a. Note that for knots these are the terms with z-exponent equal to 0. For a given link L, we will denote the coefficient of z^j in the expansion of $G_L(a,z)$ by $q_j(a)$.

Proposition 4. *Let K be a knot. Then $q_0'(1)=0$, and for $n \geq 2$ the n^{th} derivative of $q_0(a)$ evaluated at 1 is a multiple of C, where*

$$C = \begin{cases} 4n(n-1) & \text{if } n \equiv 2,3 \\ 2n(n-1) & \text{if } n \equiv 0,1 \end{cases} \pmod 4.$$

Proof. Write $G_K(a,z) = (a^4 - 2a^2 + 1 - a^2z^2)g(a,z) + 1$. Recall that for knots, the lowest power of z in $G_K(a,z)$ is 0. Thus, letting $g_0(a)$ be the coefficient of z^0 in the z-expansion of $g(a,z)$, we may write

$$q_0(a) = (a^4 - 2a^2 + 1)g_0(a) + 1.$$

Thus $q_0(1) = 1$, $q_0'(1) = 0$, $q_0''(1) = 8g_0(1)$, and $q_0'''(1) = 24\big(g_0(1) + g_0'(1)\big)$. For $n \geq 4$, we have

$$q_0^{(n)}(1) = 8\binom{n}{2}g_0^{(n-2)}(1) + 24\binom{n}{3}g_0^{(n-3)}(1) + 24\binom{n}{4}g_0^{(n-4)}(1).$$

Thus $q_0^{(n)}(1)$ is a multiple of

$$C = \gcd\big(4n(n-1), 4n(n-1)(n-2), n(n-1)(n-2)(n-3)\big).$$

If $n-2$ or $n-3$ is a multiple of 4, then $C = 4n(n-1)$. Otherwise, either n or $n-1$ is a multiple of 4, so either $n-2$ or $n-3$ is congruent to 2 (mod 4), giving us $C = 2n(n-1)$. □

Recall that the Jones polynomial can be recovered from the Homflypt polynomial by the substitutions $a = A^4$ and $z = A^2 - A^{-2}$. That is,

$$V_L(A) = G_L(A^4, A^2 - A^{-2}).$$

The first author shows in [3] that for knots K_1 and K_2, the difference of their Jones polynomials is always a multiple of $A^{16} - A^{12} - A^4 + 1$. To obtain that result it was insufficient to consider diagrams that differed by crossing changes; it was necessary to consider Δ-moves. We will revisit Δ-moves in Section 4, but we note here that the substitution for a and z above provides us with a new proof of the result for Jones polynomials.

Proposition 5. *Let K_1 and K_2 be knots. Then $V_{K_1}(A) - V_{K_2}(A)$ is a multiple of $A^{16} - A^{12} - A^4 + 1$.*

Proof. Substituting $a = A^4$ and $z = A^2 - A^{-2}$ into (1), we obtain $P_h(a,z) = A^{16} - A^{12} - A^4 + 1$. Since Homflypt polynomials of knots have no terms with negative z exponents, the result follows. □

Note that for links with two or more components, Proposition 5 in general fails. This is because with the same substitution for a and z,

$$\frac{P_h(a,z)}{z} = \frac{A^{16} - A^{12} - A^4 + 1}{A^2 - A^{-2}} = A^2(A^{12} - 1).$$

Proposition 5 was used in [4] to show that the Jones polynomial of a knot is never a product of another Jones polynomial with a nontrivial monomial. Here we show the analogous fact for Homflypt polynomials. It follows as an immediate corollary that a Homflypt polynomial cannot be a nontrivial monomial.

Theorem 6. *Let L_1 and L_2 be k-component links ($k \geq 1$). If $G_{L_2} = ra^n z^m G_{L_1}$, then $r = 1$ and $n = m = 0$.*

Proof. Suppose $G_{L_2} = ra^n z^m G_{L_1}$. By Theorem 1 we may write

$$G_{L_2} - G_{L_1} = (a^4 - 2a^2 + 1 - a^2 z^2) g(a,z),$$

and so

$$G_{L_1}(ra^n z^m - 1) = (a^4 - 2a^2 + 1 - a^2 z^2) g(a,z) \tag{2}$$

for all a and z. In particular, if $z = 1$ and $a = \frac{1+\sqrt{5}}{2}$, then $a^4 - 2a^2 + 1 - a^2 z^2 = 0$, and by Corollary 3, we have $G_{L_1} = 1$. Thus $r\left(\frac{1+\sqrt{5}}{2}\right)^n = 1$. It follows (e.g., from the binomial theorem) that $n = 0$, and so $r = 1$. Substituting into (2), we obtain

$$G_{L_1}(z^m - 1) = (a^4 - a^2 z^2 - 2a^2 + 1) g(a,z).$$

Letting $z = 0$ and $a = 1$ produces $0^m = 1$, since $G_{L_1} = 1$ again by Corollary 3. Thus, $m = 0$. $\square$

Corollary 7. *The Homflypt polynomial of a knot cannot be a nontrivial monomial.*

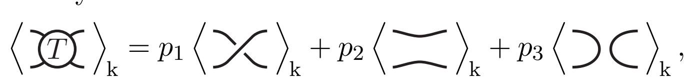
Proof. Since the Homflypt polynomial of the unknot is 1, this follows from Theorem 6, by letting L_1 be the unknot. $\square$

4. Local Moves and Kauffman Polynomials

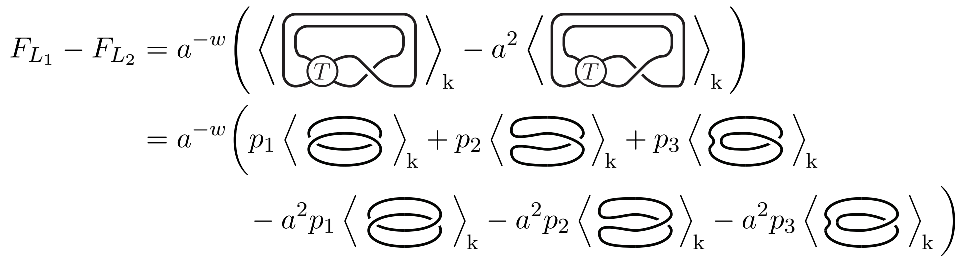
In this section we prove statements about the Kauffman polynomial analogous to those in Section 3. We begin by considering links whose diagrams differ by crossing changes as before. However, for knots, or links with an additional property, we can prove a much stronger theorem.

Theorem 8. *Let L_1 and L_2 be n-component links. Then $F_{L_1} - F_{L_2}$ is a multiple of $a^2 + 1$.*

Proof. The proof is analogous to that of Theorem 1. Observe that for any tangle T, we may write



where p_1, p_2 and p_3 are polynomials in a and z. Now consider two n-component links L_1 and L_2 that differ by a crossing change. Let w and $w-2$ be their respective writhes. (The result is unchanged if L_2 has the greater writhe.) We compute



$$
\begin{aligned}
&= a^{-w}\big(\lambda p_1 + a p_2 + a^{-1} p_3 \\
&\qquad - a^2 p_1 (az + a^{-1}z - az^{-1} - a^{-1}z^{-1} + 1) - a p_2 - a^3 p_3\big) \\
&= a^{-w}\big(p_1(az^{-1} + a^{-1}z^{-1} - 1 - a^3 z - az + a^3 z^{-1} + az^{-1} - a^2) \\
&\qquad + p_3(a^{-1} - a^3)\big) \\
&= -a^{-w-1}(a^2+1)\big(p_1(a^2 z + a - a^2 z^{-1} - z^{-1}) + p_3(a^2-1)\big)
\end{aligned}
$$

as desired. □

When links have the additional property of having the same pairwise linking numbers, we obtain a stronger result. The following definition is taken from [10]:

Definition 9. Two oriented and ordered links $L = K_1 \cup K_2 \cup \cdots \cup K_n$ and $L' = K'_1 \cup K'_2 \cup \cdots \cup K'_m$ are said to be *link-homologous* if and only if $n = m$ and $\mathrm{lk}(K_i, K_j) = \mathrm{lk}(K'_i, K'_j)$ for each pair i and j, $(1 \le i < j \le n)$.

Note that all knots are trivially link-homologous.

Theorem 10 (Analog of Theorem 1)**.** *Let L_1 and L_2 be link-homologous. Then $F_{L_1} - F_{L_2}$ is a multiple of $(a^2+1)(a^2+az+1)$.*

Proof. It is shown in [10] that two links are link-homologous if and only if they are Δ-equivalent. We will show that if two links differ by a Δ-move then the difference of their Kauffman polynomials is a multiple of $(a^2+1)(a^2+az+1)$. The result follows.

Consider links L_1 and L_2 that differ by a Δ-move, as in Figure 3. Using

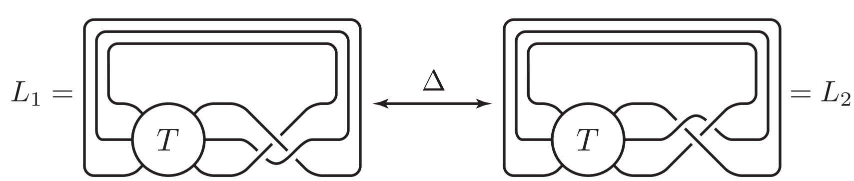
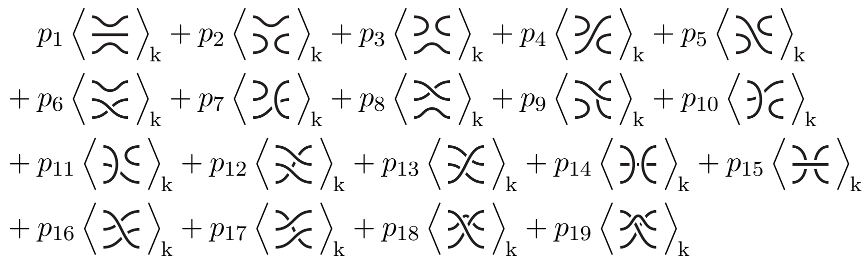


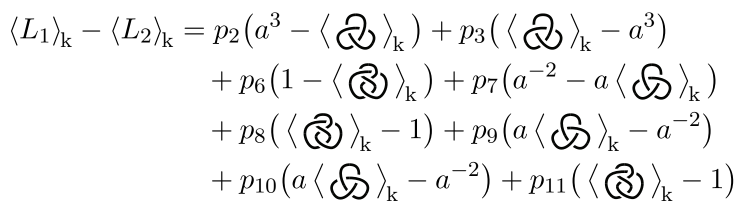
FIGURE 3. Two links that differ by a Δ-move.

the linear relation in the definition of the Kauffman polynomial, we may write $\langle L\rangle_{\mathrm{k}}$ in terms of brackets of simpler links. Specifically, for any 3-tangle T we may write  as a sum



where each p_i is a polynomial in a and z.

We show that $F_{L_1} - F_{L_2}$ has the required factor. Since the writhes of L_1 and L_2 are equal, it suffices to compute $\langle L_1\rangle_{\mathrm{k}} - \langle L_2\rangle_{\mathrm{k}}$. We have



$$
\begin{aligned}
&= a^{-2}(a^2+1)(a^2+az+1)\big(p_2(a-z) + p_3(-a+z) \\
&\quad + p_6(-z^2+1) + p_7(-az+1) + p_8(z^2-1) \\
&\quad + p_9(az-1) + p_{10}(az-1) + p_{11}(z^2-1)\big)
\end{aligned}
$$

which has the required factors. $\square$

The polynomial $(a^2+1)(a^2+az+1)$ will be denoted by $P_k(a,z)$:

$$P_k(a,z) = a^4 + a^3z + 2a^2 + az + 1 = (a^2+1)(a^2+az+1).$$

Note that $P_k(a,z)$ is maximal in the same sense as $P_h(a,z)$. We can see this since

$$F_{\text{trefoil}} - F_{\text{unknot}} = a^{-5}(z-a)P_k(a,z)$$

and

$$F_{\text{figure-8}} - F_{\text{unknot}} = a^{-2}(z^2-1)P_k(a,z).$$

Corollary 11 (Analog of Corollary 2)**.** *The Kauffman polynomial of a knot has the general form $P_k(a,z)g(a,z)+1$.*

Proof. This follows immediately from Theorem 10 since the Kauffman polynomial of the unknot is 1. $\square$

Corollary 12 (Analog of Corollary 3)**.** *For any knot K, if $a = \pm i$ or $z = -\frac{a^2+1}{a}$, then $F_K(a,z) = 1$.*

Proof. These are the zeros of $P_k(a,z)$. The result follows from the general form in Corollary 11. $\square$

Analogous to Proposition 4, we may write any Kauffman polynomial as a z-expansion, where the coefficient of z^j is denoted $p_j(a)$. For knots, the derivatives of p_j evaluated at 1 may be calculated using Corollary 11.

Proposition 13 (Analog of Proposition 4)**.** *Let K be a knot, and let $p_0(a)$ be the terms of F_K with z-exponent equal to 0. Then $p_0^{(n)}(1) \equiv 0 \pmod 4$, for $n \geq 1$.*

Proof. Write $F_K(a,z) = \big((a^4+2a^2+1)+z(a^3+1)\big)g(a,z)+1$. Then letting $g_0(a)$ be the terms of $g(a,z)$ with z-exponent equal to 0, we have

$$p_0(a) = (a^4+2a^2+1)g_0(a)+1.$$

Thus $p_0'(a) = (a^4+2a^2+1)g_0'(a)+(4a^3+4a)g_0(a)$. Subsequent derivatives may be computed directly. In general we have

$$\begin{aligned} p_0^{(n)}(1) = {} & \binom{n}{0}4g_0^{(n)}(1) + \binom{n}{1}8g_0^{(n-1)}(1) + \binom{n}{2}16g_0^{(n-2)}(1) \\ & + \binom{n}{3}24g_0^{(n-3)}(1) + \binom{n}{4}24g_0^{(n-4)}(1), \end{aligned}$$

with all subsequent terms in the expansion equal 0. □

Theorem 14 (Analog of Theorem 6)**.** *Let K_1 and K_2 be knots. Suppose $F_{K_1}(a,z) = F_{K_2}(a,z)(ra^nz^m)$. Then $r=1$ and $n=m=0$.*

Proof. Suppose $F_{K_2} = ra^nz^mF_{K_1}$. By Theorem 10 we may write

$$F_{K_2} - F_{K_1} = (a^2+1)(a^2+az+1)g(a,z),$$

and so

$$F_{K_1}(ra^nz^m - 1) = (a^2+1)(a^2+az+1)g(a,z) \tag{3}$$

for all a and z. Setting $a=1$ and $z=-2$ satisfies the conditions of Corollary 12, so $F_{K_1}=1$, and we have $r(-2)^m = 1$. Thus $r=1$ and $m=0$ (since $m \geq 0$ for knots). Substituting into (3), we obtain

$$F_{K_1}(a^n-1) = (a^2+1)(a^2+az+1)g(a,z).$$

Letting $a=2$ and $z=-\frac{5}{2}$ gives us $F_{K_1}=1$ by Corollary 12, thus $2^n=1$, and $n=0$. □

Corollary 15 (Analog of Corollary 7)**.** *A Kauffman polynomial cannot be a nontrivial monomial.*

Proof. Let K_2 be the unknot in Theorem 14. Since the Kauffman polynomial of the unknot is 1, it follows that if $F_{K_1}(a,z) = ra^nz^m$ then $r=1$ and $n=m=0$. □

References

[1] P. Drube and P. Pongtanapaisan. Oriented Local Moves and Divisibility of the Jones Polynomial. *J. Knot Theory Ramifications*, to appear.

[2] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millet and A. Ocneanu. A New Polynomial Invariant of Knots and Links. *Bull. Amer. Math. Soc. (N.S.)*, 12(2):239–246, 1985.

[3] S. Ganzell. Local Moves and Restrictions on the Jones Polynomial. *J. Knot Theory Ramifications*, 23(2):1450011, 8, 2014.

[4] S. Ganzell, J. Huffman, L. Mavrakis, K. Tademy and G. Walker. Unoriented Links and the Jones Polynomial. *Involve*, 12-8, 1357–1367, 2019.

[5] V. F. R. Jones. A Polynomial Invariant for Knots via Von Neumann Algebras. *Bull. Amer. Math. Soc. (N.S.)*, 12(1):103–111, 1985.

[6] V. F. R. Jones. Hecke Algebra Representations of Braid Groups and Link Polynomials. *Ann. of Math.* (2) 126, 335–388, 1987.

[7] L. H. Kauffman, On Knots. *Ann. of Math. Studies*, **115**, Princeton Univ. Press, 1987.

[8] L. H. Kauffman, New Invariants in the Theory of Knots. *Amer. Math. Mon.*, 195–242, March, 1988.

[9] W. B. R. Lickorish and K. Millet. A Polynomial of Oriented Links. *Topology* 26(1):107–141, 1987.

[10] H. Murakami and Y. Nakanishi. On a Certain Move Generating Link-Homology. *Math. Ann.* 284: 75–89, 1989.

[11] R. Nikkuni. C_n-moves and the Difference of Jones Polynomials for Links. *J. Knot Theory Ramifications*, 26(5):1750029, 8, 2017.

Department of Mathematics and Computer Science, St. Mary's College of Maryland, 18952 E. Fisher Rd., St. Mary's City, MD 20686
E-mail address: `sganzell@smcm.edu`

1001 E. University Ave., SU Box 6642, Georgetown, TX 78626
E-mail address: `gonzal20@southwestern.edu`

30 Meadow Lane, Huntington, WV 25704
E-mail address: `marcum248@live.marshall.edu`

PO Box 427, Univ. of Dallas, 1845 E. Northgate Dr., Irving, TX 75062
E-mail address: `nryalls@udallas.edu`

16800 Point Lookout Rd., Campus Center #2142, St. Mary's City, MD 20686
E-mail address: `mcsantos@smcm.edu`