KNOTS WITH COMPOSITE COLORS

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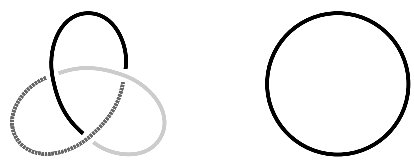
Given descriptions of two knots, how can we tell if the two knots are the same or different? This is in some sense the fundamental question of knot theory. The question applies to *links* as well as knots. Typically we just consider a knot to be a link that has only one component. In the 1950s, Fox [2] introduced the notion of a link being n-colorable, where n is any integer greater than 1. If one diagram is n-colorable and another isn't, then the two diagrams must represent different links. This is probably the easiest way to prove that the trefoil is in fact knotted—the trefoil is 3-colorable but the unknot isn't.



3-colorable. Not 3-colorable.

FIGURE 1. Different knots!

We say that a diagram of a link is *n-colored* if each arc is labeled with "colors" $0, 1, \ldots, n-1$, satisfying two conditions:

(1) More than one color is used. A coloring satisfying this condition is said to be *nontrivial*. Note that it is *not* required that all n colors are used. A color may be used more than once.
(2) At each crossing, if the arcs are labeled x, y and z, with the z being the overcrossing, then $x + y \equiv 2z \pmod{n}$.

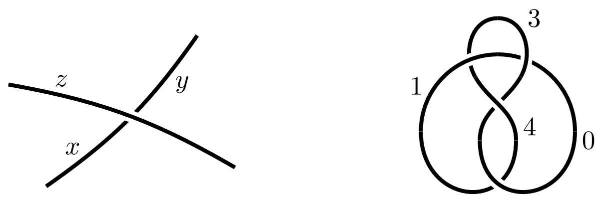


FIGURE 2. Labeling the arcs.

Let's look at an example. In Figure 2, the figure-8 knot is 5-colored: more than one color is used, and we can check condition (2) at each crossing. For example, at the crossing in the center, the arc going over is labeled 4, while the two under-arcs are labeled 3 and 0. We then verify that $3 + 0 \equiv 2 \cdot 4 \pmod{5}$.

It is a standard exercise to show that n-colorability is preserved by Reidemeister moves. See, for example [7, 5]. Since we can get from any diagram of a link to any other diagram of that link using Reidemeister moves, it follows that n-colorability is an *invariant*. That is, if one diagram of the link L is n-colorable, then every diagram of L is n-colorable. Thus we say that the figure-8 knot is 5-colorable; we don't need to refer to any specific diagram of that knot.

Is the figure-8 knot 3-colorable? 4-colorable? Or more generally, for which n is the figure-8 knot n-colorable? The usual approach to these questions is to set up a system of equations. Draw a diagram of the figure-8 (it doesn't matter which diagram) and label the arcs with unknowns representing colors. In Figure 3 the colors are a, b, c and d.

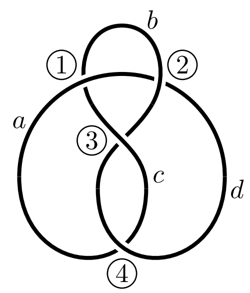


FIGURE 3. Coloring the figure-8

There are 4 crossings that need to satisfy condition (2). Thus we get a system of equations:

$$
\begin{aligned}
&(1) \quad b + c \equiv 2a \pmod{n}\\
&(2) \quad a + d \equiv 2b \pmod{n}\\
&(3) \quad b + d \equiv 2c \pmod{n}\\
&(4) \quad a + c \equiv 2d \pmod{n}
\end{aligned}
$$

Determining the values of n for which the figure-8 is n-colorable is now a matter of finding the solutions to a homogeneous system of linear equations (mod n). Specifically, we want to know for which values of n the system

$$
\begin{array}{rcrcrcrcl}
-2a & + & b & + & c & & & = & 0\\
a & - & 2b & + & & + & d & = & 0\\
 & & b & - & 2c & + & d & = & 0\\
a & & & + & c & - & 2d & = & 0
\end{array}
$$

has solutions where a, b, c and d are not all equal.

There are two observations we should make at this point. First, one of the equations is redundant: adding the first three equations gives us the negative of the fourth. So we can eliminate the fourth (or any one) equation. Second, if (a, b, c, d) is a solution to the system, then so is $(a + 1, b + 1, c + 1, d + 1)$. Can you see why? Thus we can keep adding 1 to a, b, c and d (mod n) until one of the unknowns equals 0.

So in determining which values of n give us a valid coloring, there is no loss of generality in assuming one of the unknowns, say d, equals 0. In other words, we are looking for a solution to the system

$$\begin{array}{rcrcrcl} -2a & + & b & + & c & = & 0 \\ a & - & 2b & & & = & 0 \\ & & b & - & 2c & = & 0 \end{array}$$

where a, b and c do not all equal 0 (since we are taking $d = 0$). That is, we are seeking a nontrivial solution (in the linear algebra sense) to a homogeneous system of linear equations.

Typically we are concerned with the case where n is prime. There are two reasons for this. First, to determine whether the above system has a nontrivial solution modulo a prime is easier; we need only take the determinant of the coefficient matrix. The determinant is zero (mod n) if and only if the diagram is n-colorable. In our example,

$$\det \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix} = -5,$$

so the figure-8 is n-colorable (for prime n) if and only if $n = 5$.

Second, if the link L is n-colorable then L is also nm-colorable for any positive integer m. To see this, we need only multiply each label by m. You should verify that the resulting labeling is indeed a valid nm-coloring.

On the other hand, if L is nm-colorable, then L is not necessarily n-colorable. For example, the figure-8 is 15-colorable (since it is 5-colorable), but it is not 3-colorable. It is natural to ask whether it is possible for a knot to be nm-colorable, but neither n- nor m-colorable.

If the student has already had a course in group theory, the question can be answered as follows: For a given diagram, the set of all colorings satisfying only condition (2) forms a group under addition of labels, where the identity is the (trivial) coloring of all zeros [9, 10]. This group is finite abelian, hence a direct sum of its cyclic factors. The solution follows, though dealing with the cases when m and n are not coprime is tedious.

But many undergraduate knot theory classes do not require abstract algebra as a prerequisite. The first author taught such a class and posed this question as a homework problem. The second author provided the following solution.

Question: If a link L is nm-colorable for integers $n, m \geq 2$, must L either be n-colorable or m-colorable?

Solution: Yes. Suppose without loss of generality that a diagram of L has been nm-colored with at least one arc labeled 0. There are two possibilities:

Case 1: All arc labels are multiples of n. Then we may produce an m-coloring by dividing each label by n, since

$$nx + ny - 2nz \equiv 0 \pmod{nm} \text{ implies } x + y - 2z \equiv 0 \pmod{m}.$$

Case 2: There is an arc labeled b, where b is not a multiple of n. (In particular, b is not 0.) At each crossing we have $x + y - 2z \equiv 0 \pmod{nm}$, thus $x + y - 2z \equiv 0 \pmod{n}$. Reduce the labeling mod n by replacing each label $c \in \{0, 1, \ldots, nm-1\}$ with the label $\bar{c} = c \bmod n$. This is equivalent to subtracting some multiple of n from each label. Since $x + y - 2z \equiv 0 \pmod{n}$, it follows that $\bar{x} + \bar{y} - 2\bar{z} \equiv 0 \pmod{n}$. Furthermore, this n-coloring is nontrivial because the it contains both an arc labeled 0 and an arc labeled $\bar{b}$ where $\bar{b} \neq 0$.

One of the joys of knot theory is its accessibility for students with modest mathematical preparation. Knot coloring is a great topic for beginning mathematics students to start exploring the subject. Here is a research problem to get started. Suppose a knot is n-colorable. Under what conditions is there an n-coloring that omits some specific colors? For example, it was recently shown [6] that if a link is $(2k+1)$-colorable, where $2k+1$ is prime and at least 7, then you don't need the colors k, $2k-1$ and $2k$. So if the link L is 7-colorable, then you can always find a diagram of L (not necessarily the original diagram) with a 7-coloring that only uses the colors $0, 1, 2, 4$. (Actually, the special case for 7 colors was discovered first [8].)

In a different direction, we can calculate the *number* of distinct n-colorings (including the trivial ones) of a knot, which turns out to be an invariant called $\mathrm{col}_n(K)$ with many unknown properties. Also, the n-colorability invariant in this article can be generalized in many ways: the Alexander and Homflypt polynomials are both generalizations with elementary, combinatorial definitions. And although they have been studied for decades, undergraduates are still discovering new things about them [3].

Two good introductory books on knot theory with many accessible open questions are [4] and [1]. Neither requires any advanced mathematics, and both are good starting points for students (or their advisors) looking for research problems. Have fun!

Abstract. The technique of distinguishing one knot from another by coloring arcs and applying some basic modular arithmetic is part of most standard undergraduate knot theory classes. When we study n-colorability, we are usually only interested when n is a prime number. But what if n is composite? What can we say then?

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