RESEARCH STATEMENT

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1. Introduction

Many questions in number theory can be expressed in terms of understanding the structure of the rational points on some algebraic variety. My research is concerned with explicit methods in this area; it is often of interest to compute the rational points on a variety, or to explain why the set of such points is empty.

Consider a variety X over a number field k. A key idea in the study of the set $X(k)$ of rational points is the *Hasse principle*, the statement that

$$X(k_v) = \emptyset \text{ for all places } v \text{ of } k \text{ if and only if } X(k) = \emptyset.$$

Certainly the "if" direction of this implication is always true, but the converse is much more interesting, especially because it is generally computationally straightforward to determine a finite set of v outside of which $X(k_v)$ is always nonempty, and further to decide for the v in that finite set whether or not $X(k_v) = \emptyset$. (The necessary computational ingredients are: a Hasse-Weil bound for points on curves on X, and Hensel's lemma.)

If X is a smooth projective variety, the natural map $X(\mathbb{A}_k) \to \prod_v X(k_v)$ is a bijection ([17], pp. 98-99). So while $X(k) \subseteq X(\mathbb{A}_k)$, the more interesting direction of the Hasse principle is the statement that if $X(\mathbb{A}_k)$ is nonempty, then so is $X(k)$.

While the Hasse principle is true for certain classes of varieties (e.g. quadric hypersurfaces in projective space; Severi-Brauer varieties), it is false for many other classes. For instance, the curve in $\mathbb{P}^2$ defined by $3x^3 + 4y^3 + 5z^3 = 0$ and the cubic surface in $\mathbb{P}^3$ defined by $5x^3 + 9y^3 + 10z^3 + 12t^3 = 0$ have points in $\mathbb{Q}_p$ for all primes $p \leq \infty$, but no $\mathbb{Q}$-points. (See [16] and [4], respectively, for the original proofs of these results.)

In 1971, Manin proposed a way to explain many of the known counterexamples to the Hasse principle, based on the Brauer group of the variety X. The idea is to construct a certain subset $X(\mathbb{A}_k)^{\mathrm{Br}}$ such that

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}} \subseteq X(\mathbb{A}_k).$$

If one can show that $X(\mathbb{A}_k) = \emptyset$, one has a proof that $X(k)$ is empty as well, and if $X(\mathbb{A}_k) \neq \emptyset$ one says that there is a Brauer-Manin obstruction to the Hasse principle (or to rational points) on X. While there are examples of non-Brauer-Manin obstructions to the Hasse principle, it is conjectured that the Brauer-Manin obstruction is the only one for many classes of varieties; for example:

Conjecture 1.1. ([6]) If X is a smooth rationally connected geometrically integral k-variety, k a number field, and $X(\mathbb{A}_k)^{\mathrm{Br}}$ is nonempty, then $X(k)$ is nonempty.

Here we outline briefly how the set $X(\mathbb{A}_k)^{\mathrm{Br}}$ is constructed. For any scheme X we can define the Brauer group $\mathrm{Br}\, X = H^2(X_{\mathrm{et}}, \mathbb{G}_m)$; the functor $X \mapsto \mathrm{Br}\, X$ is contravariant, and $\mathrm{Br}(\mathrm{Spec}\, k) = \mathrm{Br}\, k$, the usual Brauer group of a field. A point $P \in X(K)$, K any field, induces an evaluation map $\mathcal{A} \mapsto \mathcal{A}(P)$ sending $\operatorname{Br} X$ to $\operatorname{Br} K$. Putting this together with the exact sequence from class field theory

$$0 \to \operatorname{Br}(k) \to \bigoplus_v \operatorname{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{1}$$

where the surjective map is $\sum \operatorname{inv}_v$, we define

$$X(\mathbb{A}_k)^{\mathcal{A}} = \{(P_v) \in X(\mathbb{A}_k \colon \sum_v \operatorname{inv}_v \mathcal{A}(P_v) = 0\}.$$

Define $X(\mathbb{A}_k)^{\mathrm{Br}}$ to be the intersection of the $X(\mathbb{A}_k)^{\mathcal{A}}$ for all $\mathcal{A} \in \operatorname{Br} X$.

If $X(\mathbb{A}_k)$ is nonempty, the natural map $\operatorname{Br} k \to \operatorname{Br} X$ is injective, and understanding the cokernel of this map will be essential for computing the Brauer-Manin obstruction. Let $\overline{X} = X \times_k \overline{k}$. If we define the algebraic part of the Brauer group, $\operatorname{Br}_1 X$, to be the kernel of the map $\operatorname{Br} X \to \operatorname{Br} \overline{X}$, we have an isomorphism

$$\frac{\operatorname{Br}_1(X)}{\operatorname{Br} k} \to H^1(k, \operatorname{Pic} \overline{X}) \tag{2}$$

coming from the Hochschild-Serre spectral sequence.

One of the many difficulties in explicit computation of the Brauer-Manin obstruction is writing down actual elements of $\operatorname{Br}_1 X$ in such a way that their invariants can be computed easily. The isomorphism (2) can be given relatively explicitly, but its inverse cannot-one must make explicit use of the fact, due originally to Tate, that $H^3(k, \overline{k}^*) = 0$. We do, however, have the following useful lemma (quoted for example in [18]) about certain kinds of elements of $\operatorname{Br}_1 X$.

Lemma 1.2. *Let L/k be a finite cyclic extension of degree n, $f \in k(X)^*$. Then the class of the cyclic algebra $(L/k, f)$ in $\operatorname{Br} k(X)$ is in the image of the map $\operatorname{Br} X \to \operatorname{Br} k(X)$ if and only if the divisor of f is the norm from L to k of a divisor $D \in \operatorname{Div} X_L$. Moreover, if $L = k(\sqrt[n]{c})$ and θ is a fixed nth root of unity in k, considering $(L/k, f)$ as a representative of an element of $\operatorname{Br} X$, the invariant $\operatorname{inv}_v(L/k, f)(P_v)$ equals the norm residue symbol $[c, f(P_v)]_v$, written additively as an element of $\frac{1}{n}Z/Z$ (using the choice of the nth root of unity θ).*

2. Current and previous research

2.1. **Del Pezzo surfaces.** In my 2005 thesis [9], I explored the computation of the Brauer-Manin obstruction to the Hasse principle for smooth cubic hypersurfaces in $\mathbb{P}^3$ (Del Pezzo surfaces of degree 3), and double covers of $\mathbb{P}^2$ ramified over a smooth quartic curve (Del Pezzo surfaces of degree 2). In both cases, the calculation of the group $H^1(k, \operatorname{Pic} \overline{X})$ was the first step. My first result completed the classification of the possible finite abelian groups that could appear as $H^1(k, \operatorname{Pic} \overline{X})$, where X is a Del Pezzo surface of any degree over a field k of characteristic 0. This included as special cases computations previously done by Manin for $5 \le d \le 9$ ([15], Theorem 29.3), Swinnerton-Dyer for $d = 3, 4$ ([18]), and Urabe in certain special cases for any d ([19]). The complete classification is too lengthy to give here (see [9], Theorem 1.4.1). By results in [17], this completed the classification of $H^1(k, \operatorname{Pic} \overline{X})$ for all rational surfaces X/k.

The proof of this result was computational in nature: for $d = 1, 2$, one can reduce to the computation of the cohomology group $H^1(H, \mathbb{Z}^{r+1})$, where $r = 9 - d$ and H is a subgroup of the Weyl group $W(E_r)$. While this computation might have been achievable by a brute-force list of all of the subgroups (and the cohomology package in the computer algebra system MAGMA, which was in fact used fairly extensively), the proof simplified these computations and related nontrivial elements of this H^1 to various Galois-invariant configurations of exceptional curves on these surfaces, following ideas of Swinnerton-Dyer in [18]. This was useful in the next step, writing down examples of cyclic Azumaya algebras on these Del Pezzo surfaces X (as in Lemma 1.2) and computing their invariants. Using these ideas, I obtained the following theorem:

Theorem 2.1. *The Brauer-Manin obstruction is the only one for projective cubic surfaces over Q of the form $ax^3 + by^3 + cz^3 + dt^3 = 0$, for positive integers $a, b, c, d \leq 200$.*

The proof was carried out by first sieving all such cubic surfaces, retaining only ones with points everywhere locally but no rational points of small height. After this left a manageable number of surfaces, a program I wrote in MAGMA decided whether or not there was a Brauer-Manin obstruction to rational points coming from a cyclic algebra (automating the invariant computations that arise at each place of $\mathbb{Q}$), and a more extensive search found that all the remaining surfaces with no Brauer-Manin obstruction had a rational point. This extended previous computational results in [7], which were obtained using a somewhat different algorithm for computing the obstruction.

I also carried out more extensive work on the Brauer-Manin obstruction on Del Pezzo surfaces of degree 2. In particular, I was able to completely classify elements of order 2 in $H^1(k, \operatorname{Pic} \overline{X})$, dividing them into two types. Under the assumption that $H^1(k, \operatorname{Pic} \overline{X})$ contains an element of the first type (an assumption that I showed is satisfied whenever, for instance, the equation of the surface is of the form $w^2 = f(x^2, y^2, z^2)$, f homogeneous of degree 2), I was able to give an explicit algorithm for computing the Brauer-Manin obstruction coming from the corresponding element of $(\operatorname{Br}_1 X)/(\operatorname{Br} k)$. The computation of this element made use of some classical algebraic geometry regarding the bitangents to a plane quartic, as well as some lemmas from Galois cohomology; the details can be found in the paper [10]. Recently, the classification of order-2 elements has been expanded upon by Logan in [13], who carried out similar computations for elements of the second type on certain kinds of Del Pezzo surfaces of degree 2.

Once the algorithm described above was programmed into MAGMA, I was able to prove the following theorem by searching for rational points on surfaces with no obstruction, just as in the degree-3 case.

Theorem 2.2. *The Brauer-Manin obstruction is the only one for surfaces over $\mathbb{Q}$ given by the equation*

$$w^2 = ax^4 + by^4 + cz^4, \quad a, b, c \in \mathbb{Z}, \; |a|, |b|, |c| \leq 50.$$

The same conclusion holds for

$$w^2 = ax^4 + by^4 + cz^4 + 2dx^2y^2 + 2ex^2z^2 + 2fy^2z^2, \quad a, b, c, d, e, f \in \mathbb{Z}, \; |a|, |b|, |c|, |d|, |e|, |f| \leq 5.$$

(The second result was proved recently, and has not yet been published.)

Each of these theorems provides the best-known current computational evidence for the truth of Conjecture 1.1 on Del Pezzo surfaces of degree 3 or 2.

2.2. **K3 surfaces and genus-2 curves.** Some of my current research focuses on an idea developed by van Luijk and Logan in [14]. Let C be a genus-2 curve over a number field k

with Jacobian J, and consider the well-known exact sequence

$$0 \to J(k)/2J(k) \to \mathrm{Sel}^{(2)}(k, J) \to Ш(k, J)[2] \to 0$$

The elements in $Ш(k, J)[2]$ come from the failure of the Hasse principle on certain principal homogeneous spaces of J, and much recent research (e.g. [3], [1]) has been devoted to exhibiting nontrivial elements of this group. This is part of an effort to carry out 2-descent on these curves, as in the more well-known theory in the genus-1 case.

The main idea of [14] can be expressed as follows: desingularizations of certain quotients of these principal homogeneous spaces are $K3$ surfaces V_δ ($\delta \in \mathrm{Sel}^{(2)}(k, J)$), on which we can find various explicit Brauer group elements which can be used to find counterexamples to the Hasse principle. This in turn leads to nontrivial elements of $Ш(k, J)[2]$.

The $K3$ surfaces V_δ are in fact twists of the desingularized Kummer surface of J, so $\overline{V} = V_\delta \times_k \overline{k}$ contains 32 lines, whose divisor classes generate a free subgroup of $\mathrm{Pic}\,\overline{V}$ of rank 17. Just as for Del Pezzo surfaces, knowing the intersections of, and Galois action on, these lines lets us compute the possibilities for $H^1(k, \mathrm{Pic}\,\overline{V})$. In this case, the action of the absolute Galois group G_k on the 32 lines gives a map $G_k \to A$, where A is the set of automorphisms of the lines which preserve intersections. Here A is a group of order 23040. (In the Del Pezzo case, the analogue of A is the Weyl group $W(E_r)$.)

The paper [14] uses elliptic fibrations on $K3$ surfaces as in [2] to construct nontrivial Azumaya algebras, but I have found that the cyclic-algebra approach can be made to work here as well. In the paper [8], I determine conditions on C which ensure that there is a quaternion algebra in $\mathrm{Br}_1 V$ of the type given in Lemma 1.2. This comes down to controlling the image H of the homomorphism $G_k \to A$; we look for subgroups H such that $H^1(H, \mathbb{Z}^{17}) = Z/2$ but $H^1(H_0, \mathbb{Z}^{17}) = 0$, for some index-2 subgroup H_0. This is necessary in order to have a quaternion algebra representing the nontrivial element of $(\mathrm{Br}_1 V_\delta)/(\mathrm{Br}\, k)$. (Note that if this $\mathbb{Z}^{17}$ is a proper subgroup of $\mathrm{Pic}\,\overline{V}$, the map $H^1(H, \mathbb{Z}^{17}) \to H^1(H, \mathrm{Pic}\,\overline{V})$ may be the zero map, so we will not get a nonconstant algebra via this construction. However, results in [14] show that generically the subgroup generated by the lines equals all of $\mathrm{Pic}\,\overline{V}$.)

In particular, an analysis of the subgroups of A using MAGMA shows that there are two conjugacy classes of subgroups H of order 96 and 128 which meet these qualifications (in fact, any subgroup meeting these qualifications is contained in a subgroup in one of these two classes). Assuming that the image of $G_k \to A$ lies in the subgroup of order 96, I give an algorithm that constructs the corresponding quaternion algebra. Just as it did for Del Pezzo surfaces, the construction uses our knowledge of the Picard group, and is essentially a method for constructing an L-rational divisor in an L-rational divisor class (the one coming from Lemma 1.2), where L is the quadratic extension equal to the fixed field of H_0. As in the algorithms in [11] and [9] for Del Pezzo surfaces, the critical part of the computation is solving a norm equation from a cyclic extension of a certain number field which can be solved everywhere locally, hence has a global solution by the Hasse Norm Theorem.

The last step in the construction is to search using this algorithm for a C and V_δ over $\mathbb{Q}$ for which the corresponding algebra has constant invariants at every place, summing to $1/2$. The result is the following theorem:

Theorem 2.3. *Let n be a product of primes splitting completely in a certain finite extension $K/\mathbb{Q}$. Then the Jacobian J of the curve $y^2 = n(x^2 - 5x + 1)(x^3 - 7x + 10)(x + 1)$ has $Ш(\mathbb{Q}, J)[2] = \mathbb{Z}/2 \times \mathbb{Z}/2$.*

Most of the content of the proof of this theorem is in the proof of the case $n = 1$; just as in the main theorem of [14], the statement for general n follows from an easy lemma showing that the Selmer group element δ we used in the case $n = 1$ is still an element of the Selmer group for any n given by the theorem. The local invariant computations that occur in this theorem are very delicate, and have not yet been fully automated. An ongoing project of mine is to create MAGMA programs, similar to Logan's program [12], which will take a curve C and element δ as input and decide whether there is a Brauer-Manin obstruction to the Hasse principle on V_δ.

Currently I am trying to construct examples along the same lines using the order-128 subgroup of A. Moreover, this subgroup contains the order-32 subgroup which was the generic image of $G_k \to A$ in the setting of [14]; hence, I hope to be able to explain the counterexample to the Hasse principle given in [14], and any example obtained using elliptic fibrations, with my cyclic-algebra methods as well.

2.3. **Torsion points on CM elliptic curves.** Another current project is the subject of a VIGRE research group which was run last year at the University of Georgia by Prof. Pete Clark and myself. In 2004, Clark developed an algorithm for classifying the possibilities for the torsion subgroup of a CM elliptic curve over a number field of degree d over $\mathbb{Q}$. I led an effort to write and publish MAGMA programs which carry out this computation. The data generated by these programs led to a multitude of theoretical results and conjectures, which have been written up in the preprint [5]. I am planning to write up the computational results in a companion paper later this year, along with several members of the VIGRE group.

3. Future research directions

3.1. **2-descent and Ш[2].** I intend to continue developing the theory of descent for genus-2 curves; in particular, there are connections between the method of exhibiting Ш[2] described above and methods in e.g. [3], [1], and [14].

For instance, because the curve C in Theorem 2.3 has a Weierstrass point, we can find an odd-degree model for C and apply the techniques of [3] to obtain, for any $\delta \in \mathrm{Sel}^{(2)}(\mathbb{Q}, J)$, a Del Pezzo surface W_δ of degree 4 with points everywhere locally, such that δ gives a nontrivial element of Ш$(\mathbb{Q}, J)[2]$ if W_δ has no $\mathbb{Q}$-rational point. However, as I point out in [8], Logans program [12] shows that W_δ has no Brauer-Manin obstruction, so we expect a rational point on W_δ by Conjecture 1.1. In fact, V_δ is a double cover of W_δ, so any Brauer-Manin obstruction to rational points on W_δ should correspond to a Brauer-Manin obstruction to rational points on V_δ. I intend to study further the connection between these obstructions. In particular, when W_δ has a point, Proposition 3 of [3] shows that δ can be visualized in the Jacobian of a certain genus-4 curve. I would like to see where the $K3$ surface V_δ fits into this discussion, and how it relates to this genus-4 curve.

3.2. **Is the Brauer-Manin obstruction the only one?** It is not known, even conjecturally, whether or not the Brauer-Manin obstruction is the only one to the Hasse principle on $K3$ surfaces. The surfaces V_δ give a wealth of examples on which to test the Hasse principlein particular, if such a surface has no Brauer-Manin obstruction (which is provable using the above algorithm as long as the image of $G_k \to A$ lies inside one of the subgroups H), then we expect rational points on V_δ unless the Brauer-Manin obstruction is not the only one. One idea is to make tables of such V_δ and to search for rational points whenever V_δ has no

Brauer-Manin obstruction. So far, my searches on examples of V_δ with no obstruction have generally failed to find rational points, but this is most likely because I have not searched far enough yet (deciding how far is far enough might potentially involve Batyrev-Manin-type heuristics for $K3$ surfaces).

3.3. **Computing Brauer-Manin obstructions on other types of $K3$ surfaces.** Generally speaking, as long as enough is known about the geometric Picard group, my methods should potentially lead to examples of Brauer-Manin obstructions to the Hasse principle on other $K3$ surfaces-e.g. diagonal quartics in $\mathbb{P}^3$ (as in [2]), double covers of $\mathbb{P}^2$ ramified over a sextic, and so on. The goal is to see what can be proved using cyclic algebras on such surfaces, and what implications the absence of rational points would have in these situations.

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