Two-Dimensional Image Rotation

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Abstract. We study the problem of approximating a rotation of the plane, $\alpha : R^2 \longrightarrow R^2$ $\alpha(x, y) = (x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta)$, by a bijection $\beta : Z^2 \longrightarrow Z^2$. We show by an explicit construction that one may choose β so that $\sup_{z\in Z^2} |\alpha(z) - \beta(z)| \leq \frac{1}{\sqrt{2}}\frac{1+r}{\sqrt{1+r^2}}$, where $r = \tan(\theta/2)$. The scheme is based on those invented and patented by the second author in 1994.

1 Introduction

The need for image rotation arises in many scientific and industrial settings. A typical example of the need for rotation of an image by an arbitrary angle arises in the case of digital storage of an image by digital scanning of a document, such as an engineering drawing. When the drawing is fed into the scanner, it may be slightly skewed. Accordingly, the digitized image as it is initially stored in memory will reflect the skewing of the document relative to the scanner. When the stored image is produced on the computer screen, it will be skewed correspondingly, relative to the screen. It may be desirable to rotate the image as viewed on the screen to a desired alignment with the screen. Another example of the need for rotating an image arises in presenting the image on the computer screen or in a printed document at a rotational position different from that represented by its bit-mapping in memory.

The obvious way to rotate an image is to store the original image in one area of memory and transfer each dot from one image to the other. The problem with this method is that, as the image is rotated, the dots from the original image will fall between the dot locations in the second area of memory. For this reason, interpolation should be applied to each dot. Using that approach, there must be a series of computations performed for each dot, which will result in low speed operation (or high cost). There is a need for a simplified high speed algorithm for rotating images.

Another problem with the naive "nearest-neighbor" type algorithms is that they are neither 1-1 nor onto. This causes difficulty in ordering the storage, (for example, a pixel could be empty, white, black, doubly black, etc.), and it causes difficulty when iterating the algorithm (the image would gradually disappear!).

It is therefore desirable to approximate nearest-neighbor type accuracy using 1-1 and onto algorithms. Such schemes exist in the image processing literature. The best seems to be Paeth's scheme [P]. Which is, however, less accurate than ours (see the end of the next section).

All the schemes for rotating images in this paper are modifications of the schemes invented and patented by the second author in 1994 [T]. These schemes are fast, simple and accurate. The mathematical theory of these schemes was first investigated in [IT].

Some old [IT] and new results are contained in this paper. The new results are of two kinds. First, we prove that for infinitely many angles our schemes are the best possible ones. Second, we investigate a selection of angles for which we can prove that our schemes are not the best possible ones. Both of these developments were indicated by numerical results presented in [IT], but required further work to prove. A more detailed summary of the paper is given at the end of the next section.

2 Mathematical Statement of the Problem

We study the problem of approximating planar rotations (about the origin)

$$\alpha : R^2 \longrightarrow R^2, \quad \alpha(0,0) = (0,0),$$

by <u>1-1</u>, <u>onto</u> mappings

$$\beta : Z^2 \longrightarrow Z^2,$$

of the integer lattice $Z^2 = \{(x,y) \in R^2 | x, y \in Z\}$. β is said to approximate α with error

$$e(\beta, \alpha) = \sup_{z \in Z^2} \{|\beta(z) - \alpha(z)|\}.$$

A <u>scheme</u> S associates to every α a β. That is, if K is the space of rotations and L is the space of 1-1, onto lattice maps, then

$$S : K \longrightarrow L.$$

The error of a scheme S is

$$E_S = \sup_{\alpha \in K} \{e(S(\alpha), \alpha)\}.$$

We seek schemes S whose error E_S is as low as possible.

In Section 5 we define a scheme T and recall that

$$E_T \leq \frac{\sqrt{2+\sqrt{2}}}{2} \approx .924 \quad . \tag{1}$$

The estimate in Equation (1) is a special case of the main theorem in [IT]. This theorem gives the error bound for rotation by an angle θ, $(0 \leq \theta \leq \frac{\pi}{4})$ for the T-Scheme to be

$$\frac{1}{\sqrt{2}} \frac{1+r}{\sqrt{1+r^2}},$$

where $r = \tan\left(\frac{\theta}{2}\right)$.

In Section 6 we discuss the known lower bounds. In particular, we recall that for every scheme S

$$E_S \geq \frac{2+\sqrt{2}}{4} \approx .853 \ .$$

We also give infinitely many examples where our T-Scheme attains the lower bound and hence, is optimal.

Finally in Section 10 we discuss cases where our schemes are not optimal. Using computer searches we have discovered a variety of subtle ways to improve these cases.

It should be mentioned that many patented schemes, S, invented before those in [T] (in particular those patented in [K]) have $E_S = \infty$.

For every angle, Paeth's [P] scheme is less accurate than our scheme and is never optimal.

$$E_{Paeth} = \frac{\sqrt{5}}{2} \approx 1.12 \ .$$

3 Notation

Let $J_R = [0, \pi/4]$ and $I_R = [0, \sqrt{2}-1]$. The schemes defined in this paper are defined for angles $\theta \in J_R$. If $\tan\frac{\theta}{2} = r$, then $\sin\theta = \frac{2r}{1+r^2}$ and $\cos\theta = \frac{1-r^2}{1+r^2}$. $\tan\frac{\theta}{2}$ maps J_R 1-1, onto I_R and, without loss of generality, it suffices to consider angles in J_R. For $r \in I_R$,

$$\alpha_r := \text{rotation by the angle } 2\tan^{-1} r.$$

Remark 1. We work in pixel coordinates (in particular, y increases as it goes down).

Remark 2. The blocks defined below <u>do</u> contain their left and upper edges, but <u>not</u> their right or lower edges (blocks do not contain their upper right corners, nor their lower left corners).

Remark 3. $[x] := Floor(x) := x - x \ mod \ 1$ (i.e. the greatest integer less than or equal to x). Note that $x - 1 < [x] \leq x < [x] + 1$.

4 The P Array

A block is a rectangle $B(x, y, w, h)$ in the plane with upper left corner at (x, y), with width w and height h. We tile the plane into an array P of blocks $B(x, y, w, h)$ described below. The array P is transformed, block by block, to a new array P' of blocks $B'(x', y', w', h')$.

Assume that $\tan\frac{\theta}{2} = r \in I_R$. Let $R = \frac{1}{r}$. The idea is that P will give a tiling which is tilted by about an angle $-\frac{\theta}{2}$ and P' will tilt by roughly $\frac{\theta}{2}$ (see

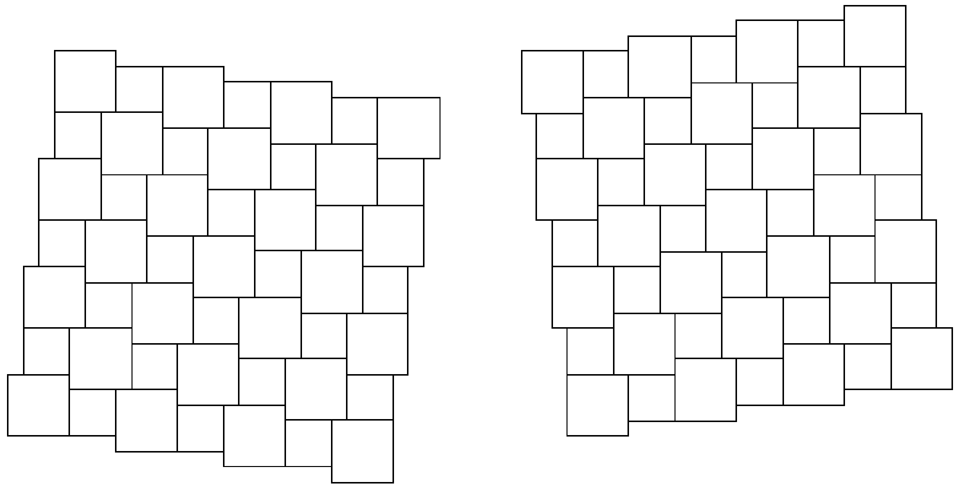

Fig. 1. P and P' Arrays for r = m/n = 1/7

Figures 1 and 2). The blocks in the arrays are labeled by integers a and b. The location of any block in the array P can be determined as follows: the upper left corner of any block $B(a,b)$ (where a+b is even) has an x-coordinate $x_\ell(a,b)$ and a y-coordinate $y_u(a,b)$ given by

$$x_\ell(a,b) = [\frac{aR - b + \zeta}{2}],$$
$$y_u(a,b) = [\frac{a + bR + \eta}{2}].$$

The choice of ζ and η is central to optimizing the scheme (see Section 6). In [T] the choice was $\zeta = 0$ and $\eta = \frac{1}{m}$ (whenever $r = \frac{m}{n}$). For our T-Scheme we choose

$$\zeta = \frac{3-R}{2}, \eta = \frac{3-R}{2}.$$

The size of the block B(a,b) is defined by determining the location of the lower and right sides. The lower and right sides have the same x and y coordinates as the adjoining sides of the contiguous blocks. The x-coordinate of the right side $x_r(a,b)$ is

$$x_r(a,b) = x_\ell(a+1,b-1) = [\frac{aR - b + R + 1 + \zeta}{2}],$$

and the y-coordinate of the lower side $y_\ell(a,b)$ is

$$y_\ell(a,b) = y_u(a+1,b+1) = [\frac{a + bR + 1 + R + \eta}{2}].$$

The bounds of the blocks having $a + b$ odd are determined from the four blocks that surround it according to the following:

$$\begin{aligned}
x_\ell(a,b) &= x_r(a-1,b) = [\frac{aR-b+1+\zeta}{2}],\\
y_u(a,b) &= y_\ell(a,b-1) = [\frac{a+bR+1+\eta}{2}],\\
x_r(a,b) &= x_\ell(a+1,b) = [\frac{aR-b+R+\zeta}{2}],\\
y_\ell(a,b) &= y_u(a,b+1) = [\frac{a+bR+R+\eta}{2}].
\end{aligned}$$

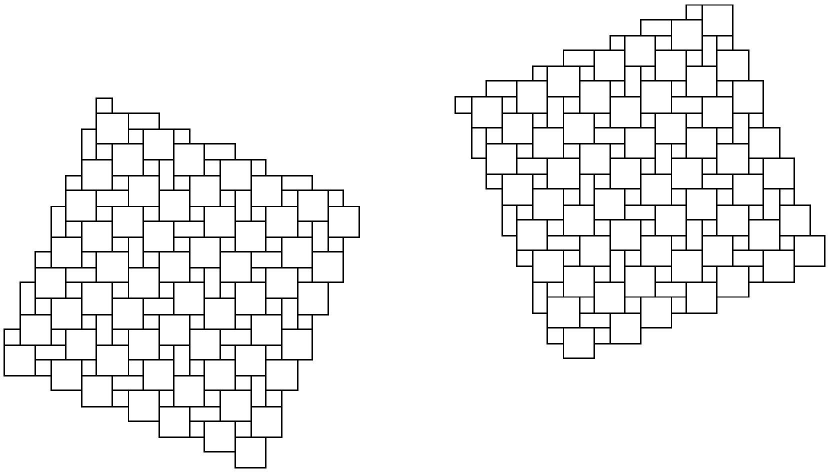

Fig. 2. P and P' Arrays for r = m/n = 2/5

Finally we have

$$\begin{aligned}
w &= x_r - x_\ell,\\
h &= y_\ell - y_u.
\end{aligned}$$

Lemma 4. *Every point in R^2 is in one and only one block.*

5 The T-Scheme

Having calculated the position and size of the blocks $B(x, y, w, h)$, the next step is to calculate the position to which the blocks are to be moved. The blocks in the new array are specified by $B'(x', y', w', h')$.

$$x' = \begin{cases} [(aR + b + \zeta')/2] & \text{if } a + b \text{ is even} \\ [(aR + b + 1 + \zeta')/2] & \text{if } a + b \text{ is odd,} \end{cases}$$

$$y' = \begin{cases} [(-a + bR + \eta')/2] & \text{if } a + b \text{ is even} \\ [(-a + bR + 1 + \eta')/2] & \text{if } a + b \text{ is odd,} \end{cases}$$

$$w' = w \ , \ h' = h.$$

For our T-Scheme we choose $\zeta' = \zeta$ and $\eta' = \eta$.

Given $r \in I_R$ the transformation of P into P' defines a map $T(\alpha_r)$ as follows: by Lemma 4 we have that every point $z = (x, y) \in Z^2$ is (uniquely) of the form

$$x = x_\ell(a, b) + u,$$
$$y = y_u(a, b) + v,$$

with $0 \le u < w(a, b)$ and $0 \le v < h(a, b)$.

$$T(\alpha_r)(x, y) := (x', y')$$

where

$$x' = x'_\ell(a, b) + u,$$
$$y' = y'_u(a, b) + v.$$

Theorem 5. *For all $r \in I_R$, $T(\alpha_r)$ is 1-1, onto, and $e(T(\alpha_r), \alpha_r) \le \frac{1}{\sqrt{2}} \frac{1+r}{\sqrt{1+r^2}}$.*

6 The Modified T-Scheme

The T-Scheme described above was chosen for ease of exposition and is not optimal. It can happen that the choice

$$\zeta = \zeta' = \eta = \eta' = \frac{3 - R}{2}$$

is not the best one. In particular, sometimes the choice

$$\zeta = \frac{5 - R}{2}, \zeta' = \frac{1 - R}{2}, \eta = \frac{3 - 3R}{2}, \eta' = \frac{3 - 3R}{2}$$

is better (and sometimes it is worse). We define the Modified T-Scheme by choosing the better of these two choices (we suspect there is never a better choice). In Figure 3 we show the error bound given by Theorem 5 and also the actual errors (using the Modified T-Scheme) for a sample of angles.

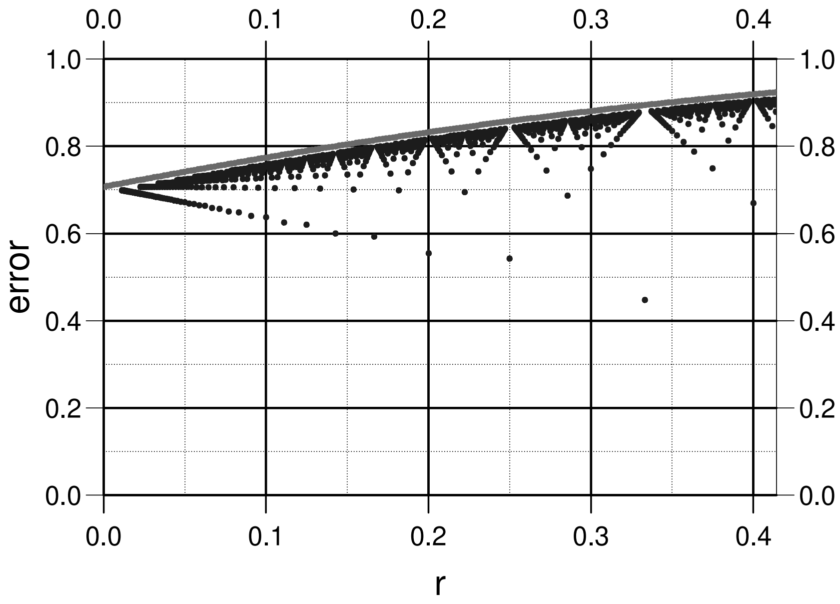

Fig. 3. Modified T-Scheme

7 Periodicity

Note that if r is rational with $r = \frac{m}{n}$ then P is periodic in the following sense:

$$\begin{aligned} x_\ell(a+2m, b) &= x_\ell(a,b) + n, \\ y_u(a+2m, b) &= y_u(a,b) + m, \\ x_\ell(a, b+2m) &= x_\ell(a,b) - m, \\ y_u(a, b+2m) &= y_u(a,b) + n. \end{aligned}$$

The $4m^2$ blocks given, as we vary from a to $a+2m$ and from b to $b+2m$, form a fundamental domain F. There are m^2+n^2 "pixels" in each fundamental domain. When computationally studying the rational case, it suffices to restrict consideration to a fundamental domain.

8 Lower Bounds

8.1 Lower Bounds for "Rational" Angles

Proposition 6. *For all S, $E_S > .787$. In fact, for all β, $e(\beta, \alpha_{``\sqrt{2}-1"}) > .787$.*

Proof. Assume not. Say β approximates rotation by $\pi/4$ with error less than .787. β must send the point $(0, 0)$ to $(0, 0)$. The point $(1, 0)$ gets sent into some square and β must send it to a corner of this square which (i) isn't already in the image of β and (ii) is within a distance .787 of the image point. Now the point $(1, 1)$ likewise, etc. This forms a simple tree of possibilities which can be checked until one obtains a contradiction. We did this for some selected angles (see Figure 4). The maximum we found (via a deeper search than shown in Figure 4) was .787 at "$\pi/4$".

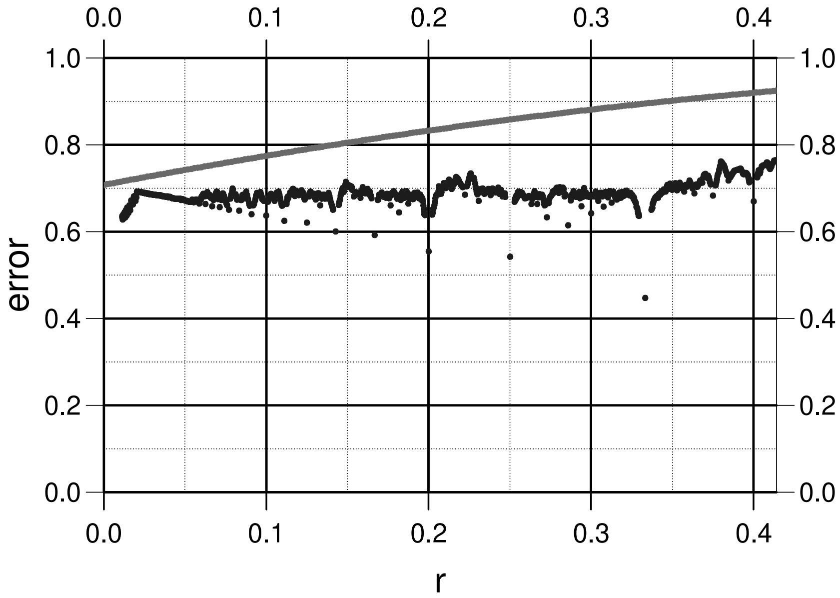

Fig. 4. Lower Bounds for "Rational" Angles

Remark 7. Note that the graphs indicate that the Modified T-Scheme is not optimal. The program actually constructs candidates for better β's; we investigated these and present our results in Section 10. In the simplest case, $\frac{2}{7}$, we have found the computer's candidate and have checked that it is indeed better.

8.2 Lower Bounds for "Irrational" Angles

As indicated by Figure 4 and by experience, the "worst" case seems to occur at $\pi/4$.

Example 8. (Due to Richard Stong.) Suppose we are trying to approximate a rotation $x \longrightarrow a(x)$ by an angle of $\pi/4$ by a bijection. There will be some point $x = (n, 0)$ such that $a(x) = (n/\sqrt{2}, n/\sqrt{2})$ is very close to $(m + 0.5, m + 0.5)$. Therefore, the images of the four points $(n, -1)$, $(n, 1)$, $(n-1, 0)$ and $(n+1, 0)$ will be approximately at $(m + 0.5 + \frac{\sqrt{2}}{2}, m + 0.5 + \frac{\sqrt{2}}{2})$, $(m + 0.5 - \frac{\sqrt{2}}{2}, m + 0.5 + \frac{\sqrt{2}}{2})$, $(m + 0.5 - \frac{\sqrt{2}}{2}, m + 0.5 - \frac{\sqrt{2}}{2})$, and $(m + 0.5 + \frac{\sqrt{2}}{2}, m + 0.5 - \frac{\sqrt{2}}{2})$. The images of these five points will all be reasonably close to the four lattice points which are vertices of the unit square containing $(m + 0.5, m + 0.5)$. However, the approximating bijection must pair one of these five points to a point not on that square. Up to symmetry it doesn't matter which, and we see that one of the five images must be moved a distance D which approximately satisfies

$$D^2 = (7 - 4\sqrt{2})/2.$$

Hence D is about 0.819.

By generalizing this example we have

Theorem 9. *For all S, $E_S \geq \frac{2+\sqrt{2}}{4} \approx .853..$ In fact, for all β, $e(\beta, \alpha_{\sqrt{2}-1}) \geq \frac{2+\sqrt{2}}{4}$.*

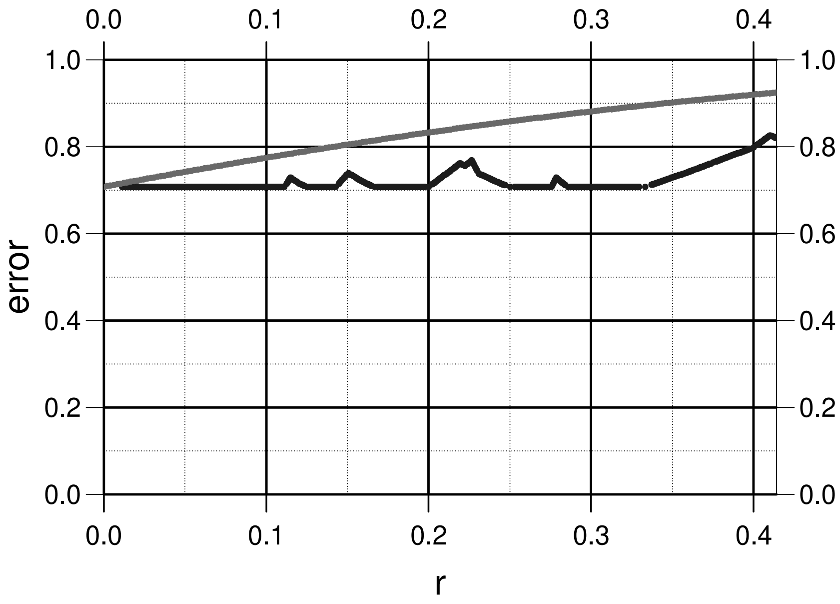

Fig. 5. Lower Bounds for "Irrational" Angles

For "irrational" angles some pixel is sent infinitesimally close to the center of a square. Modifying the computer program of Section 8.1 accordingly, we obtain Figure 5 which shows lower bounds for irrationals infinitesimally close to the rationals calculated by the computer.

9 Optimal Cases

The most important open question is: what is $M := \inf_S\{E_S\}$ where S ranges over all schemes? (We have proven that $.853 < \frac{2+\sqrt{2}}{4} \leq M \leq \sqrt{\frac{2+\sqrt{2}}{4}} < .924$.) More precisely: Does there exist a scheme (or set of schemes) S^* with $E_{S^*} = M$? Can it (or they) be described explicitly? And more generally: what is the function $M(\alpha) := \inf_{\beta \in L}\{e(\beta, \alpha)\}$? (We have proven that $Lower(\alpha_r) < M(\alpha_r) \leq \frac{1}{\sqrt{2}}\frac{1+r}{\sqrt{1+r^2}}$ where *Lower* is the lower graph in Figure 4 [Figure 5 for "irrationals"], which we've only checked for some selected angles.)

As discussed the the next section it seems quite difficult to find $M(\alpha)$ for all α. However we can now find $M(\alpha)$ for at least some α's. In this section we find countably many such α's and a few more are discussed in the next section.

Remark 10. The following theorem was motivated by comparing the lowest "line" of dots in Figure 3 with the lowest line of dots in Figure 4.

Theorem 11. *For all odd n,*

$$M(\alpha_{1/n}) = \frac{\sqrt{2}}{2}\frac{n-1}{\sqrt{n^2+1}}.$$

Proof. We prove that for these angles the modified T-scheme is optimal. In other words we prove that $e(T(\alpha_{1/n}), \alpha_{1/n}) = \frac{\sqrt{2}}{2}\frac{n-1}{\sqrt{n^2+1}}$ and that for any scheme S we have $e(S(\alpha_{1/n}), \alpha_{1/n}) \geq \frac{\sqrt{2}}{2}\frac{n-1}{\sqrt{n^2+1}}$. Given n odd the P array breaks up into squares of sides $(n+1)/2$ and $(n-1)/2$ (see for example Figure 1). The centers of these squares get sent to centers of squares in the P' array. The worst case occurs at the corner of the larger square, and a simple calculation using modular arithmetic shows that these points miss their targets by exactly $\frac{\sqrt{2}}{2}\frac{n-1}{\sqrt{n^2+1}}$. In carrying out this calculation one also sees that the targets are the nearest neighbors - which completes the proof.

Example 12. If $n = 7$ we have an error of exactly 0.6, which precisely agrees with Figures 3 and 4.

Remark 13. When n is even, we believe the modified T-Scheme is still optimal. Note that in this case the nearest neighbor algorithm is not 1-1.

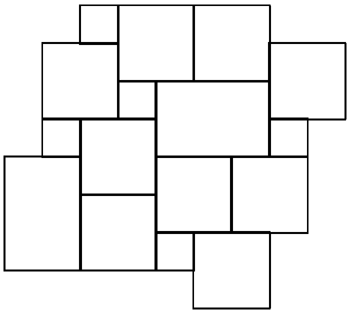

Fig. 6. Fundamental Domain for Optimal Tiling for 2/7

10 Non-optimal Cases

We believe that, except for those cases discussed above, the modified T-scheme is not optimal for the rational case. This belief is based on comparing Figures 3 and 4. On a case by case basis one can attempt to find optimal schemes. For ratios of small integers we have found optimal mappings with the computer. We have found optimal mappings for 2/5, 2/7, 3/8, 2/9, 3/10, 2/11, 2/13, 2/15, 4/19. The tiling showing the blocks which move together for the cases 2/7 and 2/9 are shown in Figures 6 and 7. These mappings are improvements over the modified T-scheme, but share some of the qualities of the T-scheme in that there is a row of tiles tilted at $\frac{1}{2}\theta$ mapped into tiles tilted at $\frac{1}{2}\theta$. Unlike the modified T-scheme the tiles do not always differ by one from their neighbors and the tiles are not always rectangular.

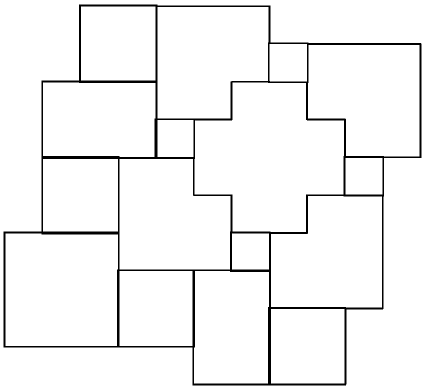

Fig. 7. Fundamental Domain for Optimal Tiling for 2/9

11 Miscellaneous Comments and Questions

- Some irrational algebraic numbers admit constructions not available for transcendental numbers. It may be that the theory is quite different for these two cases.
- What can be said about approximating rotations (about the origin)

$$\alpha : R^3 \longrightarrow R^3, \quad \alpha(0,0) = (0,0),$$

  by <u>1-1</u>, <u>onto</u> mappings

$$\beta : Z^3 \longrightarrow Z^3,$$

  of the integer lattice $Z^3 = \{(x,y,z) \in R^3 | x,y,z \in Z\}$. Or with 3 replaced by N?
- What happens if one measures the goodness of a scheme by how well it does "on the average" as opposed to "in the worst case" as we have here?
- What happens if the approximations are not required to be 1-1? Onto? If approximations are not required to be 1-1 nor onto, then clearly the "nearest neighbor" scheme is theoretically optimal, although it may be CPU intensive. An interesting related question: When is the nearest neighbor algorithm 1-1 and onto? Although not proved here, this is the case in Theorem 11.
- The circle of ideas in this paper seem to be topological and combinatorial in nature. However, there may be some meaningful (or obvious?) connection with number theory or the theory of symplectic iterators (rotations of the plane are indeed simple examples of area preserving maps, and discretizations of these have received much attention lately).
- Richard Stong observed, using Hall's Marriage Theorem, that one gets a scheme that works unless one can find a configuration as above where the rotates of k points are within r of only a total of $k-1$ points. This condition certainly seems to simplify the problem, but we cannot yet prove which "clusters" are optimal, nor which angles are "worst".

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