Integrable systems, harmonic maps and the classical theory of surfaces

M. Melko and I. Sterling

1 Harmonic Maps and Surface Theory

Many geometers in the 19th and early 20th century studied surfaces in $\mathbf{R}^3$ with particular conditions on the curvature. Examples include minimal surfaces, surfaces of constant mean curvature and surfaces of constant Gauss curvature. The typical observation was that one could introduce a special coordinate chart (i.e. curvature lines or asymptotic lines) in order to reduce the compatability conditions of the given class of surfaces to some nonlinear P.D.E. In this way, the description of a given class of surfaces is reduced to the study of the solution space of the corresponding P.D.E. The following table illustrates this correspondence. Here, the compatability conditions are given in curvature line coordinates.

Geometry	Soliton P.D.E.
constant mean curvature $H = 1/2$ constant Gauss curvature $K = 1$	$\omega_{xx} + \omega_{yy} = -\sinh\,\omega$
constant Gauss curvature $K = -1$	$\omega_{xx} - \omega_{yy} = \sin\,\omega$
minimal surfaces $H = 0$	$\omega_{xx} + \omega_{yy} = \mathrm{e}^{\omega}$

Table 1

The classical geometers were able to give detailed descriptions of those solutions that satisfy additional conditions (e.g. solutions corresponding to surfaces of rotation), but there was no methodology at hand by which one could characterize the entire space, or at least a reasonably large subset thereof. A brief discussion of their results is relegated to the third section of this paper.

The intervening period of time has seen the development of the theory of integrable systems, or soliton theory. The idea of this theory is to find a bi-Hamiltonian system (of infinite dimension) that has the given P.D.E. (or set of P.D.E.'s) as a compatability condition. This system turns out to be completely integrable, moreover, one can apply the theory of algebraic curves to construct action-angle coordinates on the corresponding phase space.

This theory, which appears to be the correct approach to describing the space of solutions corresponding to a particular class of surfaces, is outlined in the following section.

Here, we shall concern ourselves with the problem of constructing the preliminary data that is needed for the development in Section 2. This is done via harmonic maps. A more detailed account for the case that ε is the Minkowski metric may be found in [21].

In what follows, we assume that $\mathbf{R}^2$ has a metric, which we denote by ε, and we denote the corresponding Laplacian on $(\mathbf{R}^2, \varepsilon)$ by Δ. We are mainly interested in the case that ε is either the standard Euclidean metric or the standard Minkowski metric. We denote the standard metric on S^2 by δ.

A map $\psi : (\mathbf{R}^2, \varepsilon) \to (\mathrm{S}^2, \delta)$ is then defined to be *harmonic* if its *tension field* vanishes, that is

$$\tau(\psi) := \mathrm{trace}_\varepsilon(\nabla\, d\psi) = 0 \ ,$$

where ∇ denotes the connection on the vector bundle $\mathrm{T}^*(\mathbf{R}^2) \otimes \psi^*(\mathrm{TS}^2)$ induced by the metric $\varepsilon^* \otimes \delta$.

The above equation arises as the Euler-Lagrange equation for the variational problem of the energy integral

$$\mathrm{E}_{\mathrm{U}}(\psi) := \frac{1}{2} \int_{\mathrm{U}} \|\, \mathrm{d}\psi \,\|^2_{\varepsilon\otimes\delta}\, dx\, dy \ .$$

Here, U denotes an open set in $\mathbf{R}^2$ (with compact closure), and $\|\ \|_{\varepsilon\otimes\delta}$ denotes the norm on $\mathrm{T}^*(\mathbf{R}^2) \otimes \psi^*(\mathrm{TS}^2)$ induced by $\varepsilon^* \otimes \delta$.

It should be pointed out that this variational problem is only formal in the case that ε is not positive definite. Also, the harmonicity of ψ is invariant under conformal transformations (because the domian is two-dimensional).

The following well-known fact is the key to our approach:

Proposition 1.1 *A smooth map $\psi : (\mathbf{R}^2, \varepsilon) \to (\mathrm{S}^2, \delta)$ is harmonic if and only if*

$$\Delta\psi = \rho\, \psi \ , \qquad \rho : \mathbf{R}^2 \to \mathbf{R} \ . \tag{1.1}$$

The idea of the proof is simply to consider ψ as a map into $\mathbf{R}^3$, and then to compute the Euler-Lagrange equation with the constraint $\|\psi\|_\delta = 1$, see also [39].

It is natural to try to make use of the symmetric space structure of S^2 in order to put the *harmonic map equation* (1.1) in some tractable form. For that reason we introduce the following notation conventions: First, we consider $\mathrm{G} = \mathrm{S}^3$ as being the group of unit quaternions. We denote the quaternions by $\mathbf{H}$ and write $\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\}$ for the usual basis. Furthermore, we denote the Lie algebra of G by $\mathcal{G} := \mathrm{T}_1\mathrm{S}^3$, which we identify with the set of imaginary quaternions. Given these identifications, we may write the symmetric splitting of $\mathcal{G}$ corresponding to S^2 by $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$, where $\mathcal{K} := \mathbf{R}\,\mathrm{i}$, and $\mathcal{P} := \mathbf{R}\,\mathrm{j} \oplus \mathbf{R}\,\mathrm{k}$.

Since $\mathbf{R}^2$ is contractible, the pullback $\psi^*(\mathrm{S}^3)$ is necessarily a trivial circle bundle over $\mathbf{R}^2$. This is equivalent to saying that a smooth map $\psi : \mathbf{R}^2 \to \mathrm{S}^2$, always has a smooth lift $\Psi : \mathbf{R}^2 \to \mathrm{S}^3$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathrm{S}^3 \arrow[d, "\pi"] \\ \mathbf{R}^2 \arrow[ur, "\Psi"] \arrow[r, "\psi"'] & \mathrm{S}^2 \end{tikzcd} .$$

It follows that the maps ψ and Ψ are related by the identity

$$\psi = \Psi\,\mathrm{i}\,\Psi^{-1} \ . \tag{1.2}$$

We wish now to characterize equation (1.1) in terms of the derivative of the lift Ψ. Translating $d\Psi$ back to the identity from the left side gives us a $\mathcal{G}$-valued connection 1-form $\mathrm{A} := \Psi^{-1}d\Psi$. The connection A may be decomposed into its $\mathcal{K}$ and $\mathcal{P}$ parts. Specifically,

$$\mathrm{A} = \mathrm{A}'dx + \mathrm{A}''dy \ , \quad \mathrm{A}' = \mathrm{A}'_0 + \mathrm{A}'_1 \ , \quad \mathrm{A}'' = \mathrm{A}''_0 + \mathrm{A}''_1 \ , \tag{1.3}$$

where, A' , A'' are $\mathcal{G}$-valued functions, A'_0 , A''_0 are $\mathcal{K}$-valued functions, and A'_1 , A''_1 are $\mathcal{P}$-valued functions on $\mathbf{R}^2$.

The $*$-operator on the exterior algebra $\mathcal{E}^*(\mathbf{R}^2)$ induced by the metric ε extends linearly to an endomorphism of the graded Lie algebra of $\mathcal{G}$-valued exterior forms $\mathcal{E}^*(\mathbf{R}^2 , \mathcal{G}) \simeq \mathcal{E}^*(\mathbf{R}^2) \otimes \mathcal{G}$.

Definition 1.2 *The connection* A *is admissible if it satisfies the pair of equations*

$$\mathrm{dA} + \frac{1}{2}[\,\mathrm{A}\,,\,\mathrm{A}\,] = 0 \quad , \quad \mathrm{d}*\mathrm{A}_1 + [\,\mathrm{A}_0\,,\,*\mathrm{A}_1\,] = 0 \ , \tag{1.4}$$

where $\mathrm{A}_1 := \mathrm{A}'_1dx + \mathrm{A}''_1dy$ *and* $\mathrm{A}_2 := \mathrm{A}'_2dx + \mathrm{A}''_2dy$.

The first equation in (1.4) (the *integrability* or *flatness condition*) guarantees that the connection A integrates to a map Ψ. The second equation (the *harmonicity condition*) is equivalent to (1.1).

We denote the space of harmonic maps by $\mathcal{L}$, and the space of admissible connections by $\mathcal{C}$. Further, we define the gauge group to be $\mathrm{C}^\infty(\mathbf{R}^2, \mathrm{S}^1)$, where S^1 denotes the complex numbers of unit modulus. An element g in $\mathrm{C}^\infty(\mathbf{R}^2, \mathrm{S}^1)$ acts on a map Ψ, resp. a connection A by the rule

$$\Psi \mapsto \Psi \mathrm{g}^{-1} , \qquad \mathrm{A} \mapsto -d\mathrm{g}\, \mathrm{g}^{-1} + \mathrm{g}\, \mathrm{A}\, \mathrm{g}^{-1} .$$

Since a map $\psi = \Psi \mathrm{i} \Psi^{-1}$ is invariant under such gauge tranformations, one easily obtains the following:

Proposition 1.3 *There is a one-to-one correspondence between the space of harmonic maps $\mathcal{L}$ and the moduli space $\mathcal{M}$ of equivalence classes of admissible connections under the action of the gauge group.*

We have found an equivalent formulation of the harmonic map equation (1.1) in terms of the pair of equations (1.4). Now we combine the pair of equations (1.4) into the single equation (1.6) by extending the range of values of the connection A.

Let $\mathcal{G}^{\mathbf{C}} := \mathcal{G} \otimes \mathbf{C} \simeq \mathcal{SL}(\mathbf{C}^2)$ denote the complexification of $\mathcal{G}$, and set $\mathcal{K}^{\mathbf{C}} := \mathcal{K} \otimes \mathbf{C}$, $\mathcal{P}^{\mathbf{C}} := \mathcal{P} \otimes \mathbf{C}$. Then the corresponding bracket induces a complex Lie algebra structure on the ring extension $\tilde{\mathcal{G}}^{\mathbf{C}} := \mathcal{G}^{\mathbf{C}}[\lambda, \lambda^{-1}]$, where λ takes values in $\mathbf{C}_* := \mathbf{C} \setminus \{0\}$. It is easily shown that the space

$$\tilde{\mathcal{G}}^{\mathbf{C}}_\sigma = \{ \sum_a \lambda^a \xi_a \in \tilde{\mathcal{G}}^{\mathbf{C}} \mid \xi_a \in \mathcal{K}^{\mathbf{C}} \text{ if } a \text{ even, } \xi_a \in \mathcal{P}^{\mathbf{C}} \text{ if } a \text{ odd} \}$$

induced by the symmetric splitting $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ is a Lie subalgebra of $\tilde{\mathcal{G}}^{\mathbf{C}}$. Complex conjugation extends in the obvious way to $\mathcal{G}^{\mathbf{C}}$, and therefore induces a conjugation operator on the algebras $\tilde{\mathcal{G}}^{\mathbf{C}}$ and $\tilde{\mathcal{G}}^{\mathbf{C}}_\sigma$. We denote all of these conjugation operaors by $(\bar{\ })$.

We now extend A to a family of connections A^λ with values in $\tilde{\mathcal{G}}^{\mathbf{C}}_\sigma$ by the rule

$$(\mathrm{A}^\lambda)' := \mathrm{A}_0' + \lambda\, \mathrm{A}_1' , \quad (\mathrm{A}^\lambda)'' := \mathrm{A}_0'' + \lambda^{-1}\, \mathrm{A}_1'' , \quad \lambda \in \mathbf{C}_* . \tag{1.5}$$

For reasons to be explained later, λ will be referred to as the *spectral parameter*. The following fundamental observation was first due to Pohlmeyer [27]:

Proposition 1.4 A *is an element of the space of admissible connections $\mathcal{C}$ if and only if its associated loop A^λ satisfies*

$$\mathrm{dA}^\lambda + \frac{1}{2} [\mathrm{A}^\lambda , \mathrm{A}^\lambda] = 0 , \qquad \forall\ \lambda \in \mathbf{C}_* . \tag{1.6}$$

In order to recover the the geometry determined by the metric ε, we need to restrict ourselves to some real form of the complex Lie algebra $\tilde{\mathcal{G}}^{\mathbf{C}}_\sigma$. These real forms can be obtained as the fixed point sets of appropriate involutions on $\tilde{\mathcal{G}}^{\mathbf{C}}_\sigma$. If we denote the involution corresponding to the Euclidean metric by ι_{E}, and that corresponding to the Minkowski metric by ι_{M}, we have

$$\iota_{\mathrm{E}}(\xi^\lambda) = \overline{\xi^{1/\bar\lambda}} \ , \qquad \iota_{\mathrm{M}}(\xi^\lambda) = \overline{\xi^{\bar\lambda}} \ .$$

Suppose that $\xi^\lambda = \sum_a \lambda^a \xi_a$, with $\xi_a \in \mathcal{K}^{\mathbf{C}}$ if a is even, and $\xi_a \in \mathcal{P}^{\mathbf{C}}$ if a is odd. Then ξ^λ is invariant under ι_{E}, resp. ι_{M}, if $\lambda \in \mathrm{S}^1$ resp. $\lambda \in \mathbf{R}_*$, and, in both cases, $\xi_a \in \mathcal{K}$ if a is even, and $\xi_a \in \mathcal{P}$ if a is odd. In the first case, the real form of $\tilde{\mathcal{G}}_\sigma^{\mathbf{C}}$ is an example of a twisted loop algebra. In the second case, one obtains another example of a Euclidean Lie algebra. Both of these algebras can be equiped with ad-invariant inner products.

If the associated family of connections defined by (1.5) is restricted to lie in the appropriate real form of $\tilde{\mathcal{G}}_\sigma^{\mathbf{C}}$, one easily obtains the following:

Corollary 1.5 *To every harmonic map $\psi : \mathbf{R}^2 \to \mathrm{S}^2$, there is a naturally associated one-parameter family of harmonic maps ψ^λ. When ε is the Euclidean metric on $\mathbf{R}^2$, $\lambda \in \mathrm{S}^1$, and when it is the Minkowski metric, $\lambda \in \mathbf{R}_*$.*

Now we wish to consider the problem of realizing a given harmonic map as the Gauss map of some surface in Euclidean $\mathbf{R}^3$.

With appropriate regularity conditions on the harmonic map ψ, one can put a corresponding admissible connection into a normal form. This process involves a conformal transformation of $(\mathbf{R}^2, \varepsilon)$ and a gauge transformation. Each normal connection represents an equivalence class of harmonic maps under the action of the group of conformal diffeomorphisms of $(\mathbf{R}^2, \varepsilon)$.

For the case that ε is the Minkowski metric, the appropriate regularity is that ψ is "weakly regular", i. e. one assumes that the differential $d\psi$ never vanishes on the characteristic directions. This implies that the corresponding surface has a global parametrization in asymptotic coordinates. It then follows that A can be put in the form

$$\mathrm{A}' = -\mathrm{i}\,\Omega_u - \frac{1}{2}\mathrm{e}^{-\mathrm{i}\,\Omega}\,\mathrm{k}\,\mathrm{e}^{\mathrm{i}\,\Omega} \ , \qquad \mathrm{A}'' = \mathrm{i}\,\Omega_v + \frac{1}{2}\mathrm{e}^{\mathrm{i}\,\Omega}\,\mathrm{k}\,\mathrm{e}^{-\mathrm{i}\,\Omega} \ ,$$

where $\Omega = \omega/4$, and $u = x + y$ and $v = x - y$ are asymptotic coordinates. The integrability condition in (1.4) is then easily computed to be the (real) sine-Gordon equation:

$$\omega_{uv} = \omega_{xx} - \omega_{yy} = \sin\,\omega \ .$$

Similarly, if ε is the Euclidean metric, we assume that the eigenvalues of $d\psi\,|_{(x,y)}$, considered as a linear map into $\mathrm{T}_{(x,y)}\mathrm{S}^2$ are never equal. Geometrically, this condition means that the corresponding surface has no umbilics, and hence global curvature line coordinates. The normal form for A is essentially the same, and the integrability condition becomes the elliptic sinh-Gordon equation:

$$\omega_{xx} + \omega_{yy} = \sinh\,\omega \ .$$

Note that the real sine-Gordon equation and the elliptic sinh-Gordon equation are both real forms of the complex sine-Gordon equation.

Proposition 1.6 *Let ψ^λ be the family of harmonic maps associated to a harmonic map ψ, and let Ψ^λ be its lift to S^3. If ε is the Euclidean metric on $\mathbf{R}^2$, then*

$$\varphi^\lambda = \frac{d}{d\tau}|_{\tau=t}\Psi^\lambda\,(\Psi^\lambda)^{-1}\;, \qquad \lambda = \mathrm{e}^{\mathrm{i}\,\tau}$$

is a surface of constant Gauss curvature $K = +1$ in $\mathbf{R}^3$ for every t.

Similarly, if ε is the Minkowski metric on $\mathbf{R}^2$, then

$$\varphi^\lambda = \frac{d}{d\tau}|_{\tau=t}\Psi^\lambda\,(\Psi^\lambda)^{-1}\;, \qquad \lambda = \mathrm{e}^{\tau}$$

is a surface of constant negative Gauss curvature $K = -1$ in $\mathbf{R}^3$ for every t.

Furthermore, in both cases, if $\mathrm{A} = \Psi^{-1}\,d\Psi$ is in normal form, then $\mathrm{Ad}_{\Psi^\lambda}$ is a canonical frame for each surface φ^λ.

The formulas for the parametrized surface φ^λ given above are originally due to Sym [33] (for $K = -1$) and Bobenko [5] (for $K = +1$ or $H = 1/2$). (Sym has derived similar expressions for a variety of other examples.) We remind the reader that proofs for the case that ε is the Minkowski metric can be found in [21].

As noted in Table 1 the soliton P.D.E. corresponding to surfaces of constant positive Gauss curvature $K = +1$ is also the elliptic sinh-Gordon equation. Geometrically, this equivalence follows from Bonnet's Theorem that to every surface with $K = +1$ there is a parallel surface with $H = 1/2$. A surface with a linear relationship between the Gauss curvature K and the mean curvature H is called a linear Weingarten surface. Table 1 gives a complete list, up to homothety and parallelism, of the nontrivial linear Weingarten surfaces.

So far we have restricted ourselves to linear Weingarten surfaces in three-dimensional Euclidean space. In closing, we would like to mention a few other examples of interesting classes of surfaces that correspond to well-known soliton-P.D.E.'s. These are listed in Table 2, below.

A conformal harmonic immersion of $\mathbf{R}^2$ into the Lorentzian 4-Sphere is a minimal immersion. By means of Lie's sphere geometry one can construct a correspondence between such minimal immersions and Willmore surfaces in $\mathbf{R}^3$. (For a survey of Willmore surfaces, see [26].) The corresponding compatabilty conditions for the surface can be expressed in terms of the Toda system given in Table 2. The Willmore torus (Figure 9) constructed by Ferus and Pedit [16] is one of the simplest nontrivial examples constructed via this method.

If one idealizes a smoke ring as a curve in $\mathbf{R}^3$, then the surface swept out by its evolution over time (Figure 10) is another example of a class of surfaces arising from a soliton equation. In this case, the compatability condition is the nonlinear Schrödinger equation, and the corresponding surface is called a Hasimoto surface (see [17] and also [24]).

Bianchi also studied classes of surfaces whose compatability equations are of a more general character than the Weingarten surfaces discussed above. These results appear in the original version of his book, but not in the German translation. These "Bianchi surfaces" have also been studied by Antoni Sym et. al. These surfaces are discussed in more detail in [7].

Examples of such "soliton geometries" also arise in affine surface theory [34].

Geometry	Soliton P.D.E.
Willmore surfaces	$\omega_{xx} + \omega_{yy} = 2\,\mathrm{e}^{-2\omega}\cos\,\eta - 2\,\mathrm{e}^{\omega}$ $\eta_{xx} + \eta_{yy} = 2\,\mathrm{e}^{-2\omega}\sin\,\eta$
Hasimoto surfaces	$\omega_{xx} = -i\omega_t + \omega\|\omega\|^2$
Bianchi surfaces	see Bobenko's contribution [7]
affine spheres	$\omega_{xy} = \mathrm{e}^{\omega} - \mathrm{e}^{-2\omega}$

Table 2

2 Hamiltonian systems, Spectral Theory and Flat Connections.

The purpose of this section is to indicate the correspondence between the flat connectons discussed in the previous section and the theory of Hamiltonian systems.

Let's begin with the assumption that $\tilde{\mathcal{G}}$ is some Lie algebra, possibly of infinite dimension, equiped with an ad-invariant inner product $\langle\ ,\ \rangle$. In the case when $\tilde{\mathcal{G}}$ is finite dimensional, it possesses a canonical Poisson structure. If $\mathrm{C}^\infty(\tilde{\mathcal{G}})$ is used to denote the space of smooth functions on $\tilde{\mathcal{G}}$, and ∇ is used to denote the gradient on the Euclidean space $(\tilde{\mathcal{G}}, \langle\ ,\ \rangle)$, then this Poisson structure is the bilinear map $\{\ ,\ \} : \mathrm{C}^\infty(\tilde{\mathcal{G}}) \times \mathrm{C}^\infty(\tilde{\mathcal{G}}) \to \mathrm{C}^\infty(\tilde{\mathcal{G}})$ given by

$$\{f\,,\,g\}(\chi) := \langle\,\chi\,,\,[\nabla f|_\chi\,,\,\nabla g|_\chi]\,\rangle\ , \tag{2.1}$$

where $[\ ,\ ]$ denotes the bracket on $\mathcal{G}$. It is known that this bracket is nondegenerate on the adjoint orbits of $\tilde{\mathrm{G}}$ (the group corresponding to $\tilde{\mathcal{G}}$) in $\tilde{\mathcal{G}}$, making them into symplectic manifolds.

We now drop the assumption that $\tilde{\mathcal{G}}$ is finite dimensional, and suppose that $\mathrm{L} \in \mathrm{End}(\tilde{\mathcal{G}})$ is a linear operator which is self-adjoint with respect to $\langle\ ,\ \rangle$. Suppose, further, that L has a finite-dimensional invariant subspace, which we denote by $\mathbf{V}$, and that the restriction of $\xi \mapsto \mathrm{ad}^{\tilde{\mathcal{G}}}_{\mathrm{L}(\xi)}(\xi)$ to $\mathbf{V}$ defines a vector field on $\mathbf{V}$. Here, $\mathrm{ad}^{\tilde{\mathcal{G}}}$ is used to denote the adjoint representation of $\tilde{\mathcal{G}}$. The pullback of the bracket (2.1) to the subspace $\mathbf{V}$ via the inclusion map immediately gives us a finite-dimensional Poisson manifold $(\mathbf{V}, \{\ ,\ \})$. If we further define the Hamiltonian h to be the quadratic function

$$h(\chi) = \frac{1}{2}\,\langle\,\chi\,,\,\mathrm{L}(\chi)\,\rangle\ ,$$

we have that $(\mathbf{V}, \{\ ,\ \}, h)$ is a Hamiltoniansystem. Since L is self-adjoint, $\nabla\,h|_\chi = \mathrm{L}(\chi)$. Hence Hamilton's equations take the form

$$\dot{\chi} = -\mathrm{ad}^{\tilde{\mathcal{G}}}_{\mathrm{L}(\chi)}(\chi)\ . \tag{2.2}$$

Note that the conditions on L are equivalent to saying that both the gradient and skew-gradient of h are tangent to $\mathbf{V}$. It is easily shown that the solution curves to (2.2) have the form

$$\chi(t) = (\Psi(t))^{-1}\xi\Psi(t) \ , \tag{2.3}$$

where $\Psi : \mathbf{R} \to \tilde{\mathrm{G}}$ is a curve in the group $\tilde{\mathrm{G}}$ of $\tilde{\mathcal{G}}$ with $\Psi(0) = 1$, and $\chi(0) = \xi \in \mathbf{V}$ (see [21]).

Note, in particular, that the form of (2.3) implies that the spectrum of $\rho(\chi)$ is independent of the time t, where $\rho : \tilde{\mathcal{G}} \to \mathrm{End}(\mathbf{W})$ denotes a representation of $\tilde{\mathcal{G}}$ on some finite-dimensional vector space $\mathbf{W}$. Furthermore, if we complexify $\mathbf{W}$ to obtain $\mathbf{W}^{\mathbf{C}} = \mathbf{W} \otimes \mathbf{C}$, we may think of (2.3) as defining a deformation of the decomposition of $\rho(\xi)$, considered as a linear map on $\mathbf{W}^{\mathbf{C}}$, into eigenspaces.

At this point, we assume that the $\tilde{\mathcal{G}}$ is some Euclidean algebra obtained as an extension of $\mathcal{G}$, the Lie algebra of imaginary quaternions, e.g. either of the real forms of $\tilde{\mathcal{G}}_\sigma^{\mathbf{C}}$ described in Section 1. Then we have that (2.3) depends on the spectral parameter, i.e.

$$\chi^\lambda(t) = (\Psi^\lambda(t))^{-1}\xi^\lambda\Psi^\lambda(t) \ . \tag{2.4}$$

The evaluation map $\xi^\lambda \mapsto \rho(\xi^{\lambda=\lambda_0})$ gives us a representation of $\tilde{\mathcal{G}}$ into $\mathrm{End}(\mathbf{W}^{\mathbf{C}})$. In analogy with the previous remark, equation (2.4) implies that the affine algebraic curve

$$\Gamma(\xi^\lambda) := \{(\mu, \lambda) \in \mathbf{C} \times \mathbf{C}_* \mid \det(\,\mu \mathrm{I} - \rho(\chi^\lambda(t)\,) = 0\} \tag{2.5}$$

where I is the identity on $\mathcal{G}$, is independent of the time t. This curve is called the *spectral curve* associated to the loop ξ^λ. In most of the cases discussed in Section 1, $\Gamma(\xi^\lambda)$ turns out to be hyperelliptic.

By observing the behaviour of the eigenspaces of $\mathrm{ad}^{\mathcal{G}}(\chi^\lambda(t))$ in $\mathcal{G}^{\mathbf{C}}$ as a function of t, we may interpret (2.4) as a deforming complex line bundle in $\Gamma(\xi^\lambda) \times \mathcal{G}^{\mathbf{C}}$. This gives a representation of a solution to (2.3) as a line in the Jacobian of the corresponding spectral curve.

There remains the question of complete integrability of this system. Equation (2.2) is not, in general, completely integrable with the given Poisson structure.

The standard method of obtaining an integrable system is to substitute the given Lie bracket on $\tilde{\mathcal{G}}$ with an another bracket. This new bracket is given by

$$[\xi\,,\,\eta]_{\mathrm{R}} := [\mathrm{R}(\xi)\,,\,\eta] + [\xi\,,\,\mathrm{R}(\eta)]$$

where $\mathrm{R} \in \mathrm{End}(\tilde{\mathcal{G}})$ is required to satisfy the modified classical Yang-Baxter equation

$$\mathrm{R}(\,[\mathrm{R}(\xi)\,,\,\eta] + [\xi\,,\,\mathrm{R}(\eta)]\,) - [\mathrm{R}(\xi)\,,\,\mathrm{R}(\eta)] - \alpha[\xi\,,\,\eta] = 0 \ , \quad \alpha \in \mathbf{R}$$

This equation is a sufficient condition for insuring that $[\ ,\ ]_{\mathrm{R}}$ satisfies the Jacobi identity. A linear endomorphism R that satisfies the Yang-Baxter equation is called an "r-matrix" (see also [39, 9]).

We denote the canonical Poisson structure associated to $(\tilde{\mathcal{G}}, [\ ,\ ]_{\mathrm{R}})$ by $\{\ ,\ \}_{\mathrm{R}}$. This new bracket has the property that (certain) conserved quantities of $(\mathbf{V}, \{\ ,\ \}_{\mathrm{R}}, h)$ commute with respect to it. This allows us to apply Arnold's theorem on integrable systems to obtain a dynamical system on a torus (or cylinder). In fact, the branch points of $\Gamma(\xi^\lambda)$, for $\xi^\lambda \in \mathbf{V}$, together with linear coordinates on (a real subtorus of) its Jacobian $\mathrm{Jac}(\xi^\lambda)$ essentially give action-angle coordinates on $\mathbf{V}$.

In this way $\mathbf{V}$ can be given the structure of a bi-Hamiltonian system (see [22] for the definition).

By way of an example, we consider the case of surfaces with constant Gauss curvature $K = -1$ (i.e. when the metric ε on $\mathbf{R}^2$ is the Minkowski metric).

Writing $\tilde{\mathcal{G}} = \lambda^{-1}\mathcal{G}[\lambda^{-1}] \oplus \mathcal{G} \oplus \lambda\mathcal{G}[\lambda]$, we define the needed R-matrix to be $\mathrm{R} := \Pi_+ - \Pi_-$, where $\Pi_- : \tilde{\mathcal{G}} \to \lambda^{-1}\mathcal{G}[\lambda^{-1}]$ resp. $\Pi_+ : \tilde{\mathcal{G}} \to \lambda\mathcal{G}[\lambda]$ denotes the projection of a Laurent polynomial onto its tail of negative resp. positive terms. For each nonnegative integer n, we define linear operators L'_n, L''_n on $\tilde{\mathcal{G}}$ by the rules

$$\mathrm{L}'_n := \frac{1}{2}(\mathrm{R}+1) \circ \mathrm{M}_{\lambda^{1-n}}\ , \qquad \mathrm{L}''_n := \frac{1}{2}(\mathrm{R}-1) \circ \mathrm{M}_{\lambda^{n-1}}\ , \tag{2.6}$$

where $\mathrm{M}_{p(\lambda)}$ denotes multiplication by the polynomial $p(\lambda)$.

We further define a nested sequence of finite dimensional subspaces of $\tilde{\mathcal{G}}_\sigma$ by the rule

$$\tilde{\mathcal{G}}^n_\sigma = \{ \sum_{-n}^{n} \lambda^a \xi_a \in \tilde{\mathcal{G}} \mid \xi_a \in \mathcal{K} \text{ if } a \text{ even},\ \xi_a \in \mathcal{P} \text{ if } a \text{ odd} \}\ .$$

It is easily verified that the restriction of the above operators to $\tilde{\mathcal{G}}^n_\sigma$ is given by

$$\mathrm{L}'_n(\xi^\lambda) = \frac{1}{2}\xi_{n-1} + \lambda\,\xi_n\ , \qquad \mathrm{L}''_n(\xi^\lambda) = -\frac{1}{2}\xi_{1-n} - \lambda^{-1}\xi_{-n} \tag{2.7}$$

where $\xi^\lambda = \sum_a \lambda^a \xi_a \in \tilde{\mathcal{G}}^n_\sigma$. In particular, they leave $\tilde{\mathcal{G}}^n_\sigma$ invariant for n odd.

Let us denote the real form of $\tilde{\mathcal{G}}^{\mathbf{C}}_\sigma$ corresponding to ε by $\xi^\lambda \in \tilde{\mathcal{G}}^\varepsilon_\sigma$. An element $\xi^\lambda \in \tilde{\mathcal{G}}^\varepsilon_\sigma$ will be called an *admissible loop in normal form* if $\xi^\lambda = \sum_{-n}^{n} \lambda^a \xi_a$ for n odd, $\xi_{-n} = -(1/2)\,\mathrm{e}^{\mathrm{i}\,\omega_0}\,\mathrm{k}$ and $\xi_n = -(1/2)\,\mathrm{e}^{-\mathrm{i}\,\omega_0}\,\mathrm{k}$, where $\omega_0 \in \mathbf{R}$.

Recall that we defined (u, v) to be characteristic coordinates on $(\mathbf{R}^2, \varepsilon)$. For notational convenience, let ad denote the adjoint representation of $\tilde{\mathcal{G}}^\varepsilon_\sigma$.

We are now ready to state the main result of this section:

Theorem 2.1 *Let $\xi^\lambda \in \tilde{\mathcal{G}}^n_\sigma$ be an admissible loop in normal form. Then there exists a unique smooth map $\chi^\lambda : \mathbf{R}^2 \to \tilde{\mathcal{G}}^n_\sigma$ satisfying the system of Lax equations*

$$\chi_u = -\mathrm{ad}_{\mathrm{L}'_n(\chi^\lambda)}(\chi^\lambda)\ ,\ \ \chi_v = -\mathrm{ad}_{\mathrm{L}''_n(\chi^\lambda)}(\chi^\lambda)\ ,\ \ \chi^\lambda(0,0) = \xi^\lambda\ . \tag{2.8}$$

Furthermore, if $\mathrm{A}^\lambda = (\mathrm{A}^\lambda)'\,\mathrm{d}u + (\mathrm{A}^\lambda)''\,\mathrm{d}v$, where

$$(\mathrm{A}^\lambda)' := \mathrm{L}'_n(\chi^\lambda)\ , \qquad (\mathrm{A}^\lambda)'' := \mathrm{L}''_n(\chi^\lambda)\ , \tag{2.9}$$

then A^λ is the family of connections associated to a connection in normal form.

Remark: The solution χ^λ to the above system of equations also has a geometrical interpretation. It is referred to as a *polynomial Killing field* because it is a Laurent polynomial in λ and an infinitesimal symmetry of the family of connections A^λ. Furthermore, one has that the kernel of $\mathrm{ad}^{\mathcal{G}}(\chi^{\lambda=1})$ determines the direction of the axis for the corresponding surface $\varphi^{\lambda=1}$. This axis is apparent in the series of pictures at the end of this article.

3 History

Wente, in his 1984 paper [37] (Figure 4), solved the long standing problem of Hopf:

Is a constant mean curvature surface in $\mathbf{R}^3$ *(soap bubble) that is complete and compact necessarily a round sphere?*

Wente proved that there exist soap bubbles (Wente tori) with the same topological type as a torus. He had learned from Eisenhart's textbook [12] the relationship between soap bubbles and the sinh-Gordon equation discussed in Section 1. Had he read a few more pages in Eisenhart he would have seen that the classical geometers had already found the solutions he would spend years reconstructing via analytical methods [38]. These solutions play a central role in Wente's work.

In fact, in 1985, after Wente's construction, Abresch [1] rediscovered the methods of Enneper (and his school) [13] and [14], used over a century earlier. Abresch gave explicit formulas, in terms of elliptic integrals, of Wente tori. The simplifying ansatz of Enneper was the geometric condition that one family of curvature lines be planar. As mentioned in Section 1, to each constant Gauss curvature $\mathrm{K} = +1$ surface, there is a parallel surface with constant mean curvature $\mathrm{H} = 1/2$. So there are in some sense three cases: $K = -1$, $K = +1$, and $H = 1/2$. And Enneper had three students: Lenz, Bockwoldt and Voretzsch (resp!). And each wrote a doctoral dissertation [19], [8] and [35]. The dissertation of Voretzsch is the most impressive. The last six pages are tables of numbers giving the x, y, and z coordinates of a constant mean curvature surface. It is important to note that it was not known until Wente's paper in 1984 that some of these examples in fact close up to form compact soap bubbles. As early as 1982 Walter [36] attempted to construct compact examples using Enneper's method. In 1986 Abresch [2] extended Enneper's method. This second paper of Abresch played a pivotal role in the modern development of the theory.

Enneper's solutions, in the case $K = -1$, correspond to the solutions generated by (symmetric) genus two hyperelliptic spectral curves [21] (Figure 3). Solutions corresponding to genus one spectral curves arise geometrically by assuming the stronger condition that the surface is a surface of revolution. The rotationally invariant examples were discovered much earlier (in 1839) by Minding (Figure 1). The analogous constant mean curvature surfaces of revolution were found (in 1841) by Delaunay (Figure 2).

In 1883, Dobriner [10], wrote a long paper essentially repeating Enneper's 1868 work. The editor allowed (in the same issue) a scathing rebuttal by Enneper [15], criticizing Dobriner for being 15 years behind the times!

Nevertheless in 1886 Dobriner [11], solved a much harder problem posed by Enneper:

Characterize those constant Gauss curvature $K = -1$ surfaces with one family of spherical lines of curvature.

In this elegant paper, Dobriner classifies all such surfaces and derives an explicit representation of such surfaces by means of theta functions. Dobriner's solutions correspond to the solutions generated by (symmetic) genus three hyperelliptic spectral curves [21],[24] (Figures 7 and 8).

An excellent historical survey of surfaces of constant curvature is Reckziegel [28]. The earlier works of Bonnet, Dini and Joachimsthal are discussed. Also discussed are the examples of Kuen [18] (Figure 5) and Sievert [29] (Figure 6) which correspond to simple solitons and are given explicitly via trigonometric functions. One of Kuen's beautiful surfaces graces the cover of Gerd Fischer's book (see [28]).

The most detailed and comprehensive work from the classical perspective is that of Steuerwald [32]. This paper also gives a complete discussion of Bäcklund's transformation and it's relationship to Enneper surfaces.

Recall that the sine-Gordon equation, in characteristic coordinates, is given by

$$\partial_u \partial_v \omega = \sin\, \omega \ ,$$

where ω is a real-valued function on the plane. In [20], Lie observed that if ω is a solution to this equation, then $\tilde{\omega}\,(u, v) := \omega\,(\lambda u, \lambda^{-1} v)$ is also a solution for all nonzero real numbers λ. By virtue of the fundamental theory of surfaces, there is a $K = -1$ surface associated to each $\tilde{\omega}$. This collection of surfaces, constructed via Sym's formula in Section 1 is the *associated family* of the surface corresponding to ω. The parameter λ plays the role of the spectral parameter in the soliton methods described in this paper.

Some of the standard references for surface theory deal with the work of Enneper and Dobriner. These include Darboux and Eisenhart. In 1893 Adam (probably Darboux's student) also studied Enneper surfaces [3]. The Italian edition of Bianchi's textbook [4] is more inclusive than the German translation. For example, Bianchi's multisoliton like surfaces can be found there (Figure 11 and also [30], [31]).

A classification of constant mean curvature tori using soliton theory is given in Pinkall-Sterling [23]. These surfaces can be generated by theta functions using Bobenko's representation [6].

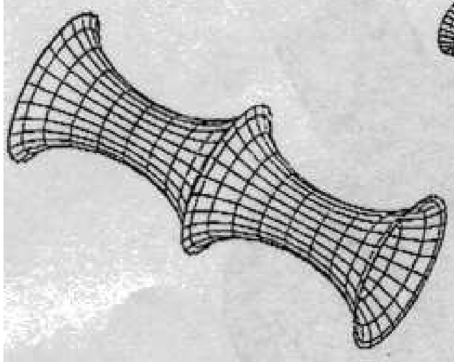

Fig. 1 Minding, $K = -1$

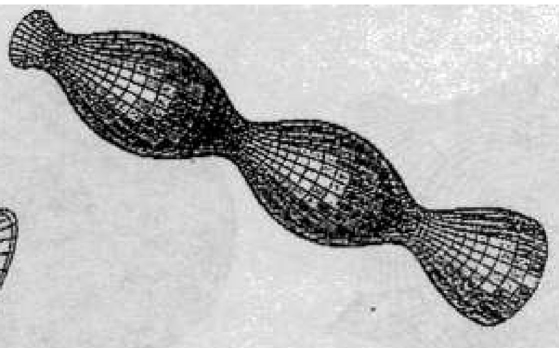

Fig. 2 Delaunay, $H = \frac{1}{2}$

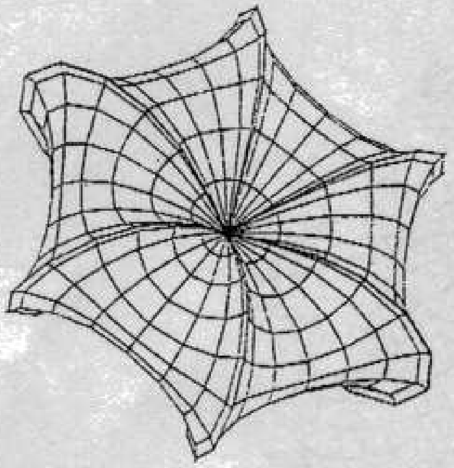

Fig. 3 Enneper, $K = -1$

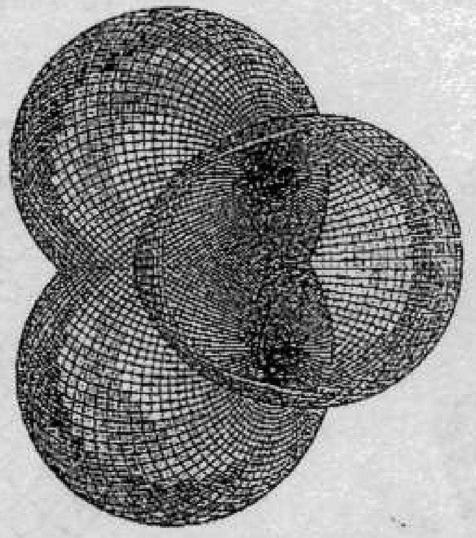

Fig. 4 Wente, $H = \frac{1}{2}$

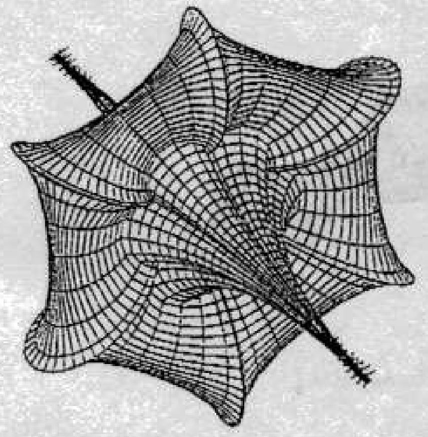

Fig. 5 Kuen, $K = -1$

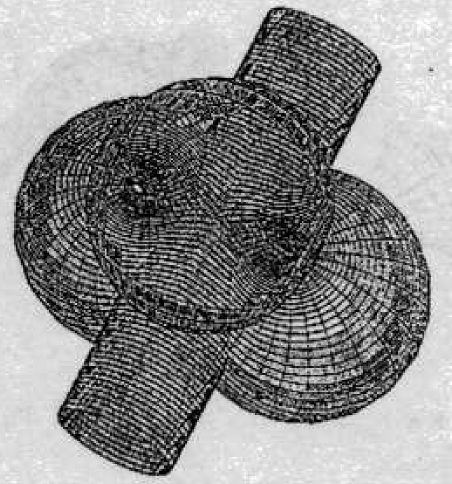

Fig. 6 Sievert, $H = \frac{1}{2}$

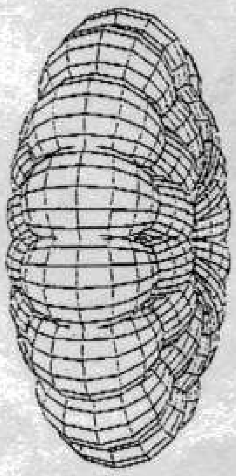

Fig. 7 Dobriner, $K = +1$

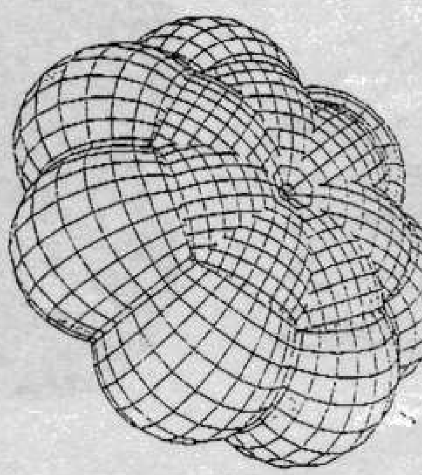

Fig. 8 Dobriner, $H = \frac{1}{2}$

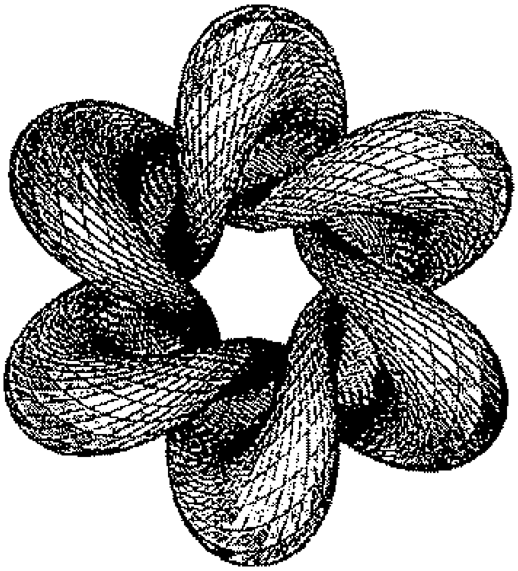

Fig. 9 Ferus-Pedit. Willmore

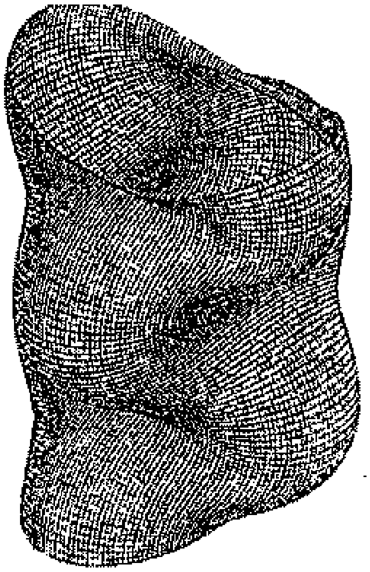

Fig. 10 Pinkall-Sterling, Hasimoto

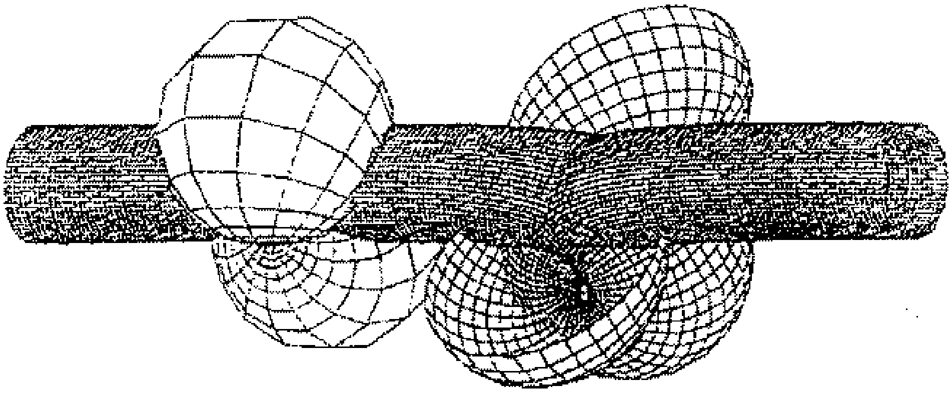

Fig. 11 Bianchi, Multibubbleton

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