Visualizing Nonlinear Electrodynamics

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Summary. Visualizing phenomena in high dimensions requires a combination of equivariant mathematics and computer graphics. This paper applies the method of equivariant geometry, which involves Lie Groups and PDE's, to the study nonlinear electrodynamics. For the difficult calculations and for the graphics we used Maple.

1. Introduction

The motivation for geometric relativization arises from physics. In a series of papers [3, 4, 5, 6, 7], the first author has developed a model for nonlinear electrodynamics by a relativization process which mimics the relativization of Newtonian mechanics into special relativity. This paper studies the moduli space of solutions to the central PDE in this theory.

The relativization of Newtonian mechanics and Maxwell's electrodynamics are the first two relativizations in an infinite series of relativizations. Preliminary investigations into the third relativization yield PDE's which reduce in special cases to the Klein-Gordon PDE and other interesting PDE's. The fourth relativization becomes very involved and has not yet been studied.

The relativization of Maxwell's electrodynamics, which is the case studied in this paper, was partially motivated by the breakdown of classical electrodynamics (and its quantum version) in the calculation of the self-energy of a point charge. It is classically calculated to be infinity. Born and Infeld [1], in the 30's, were the first to suggest non-linear versions of Maxwell's equations in order to correct this. A very readable discussion can be found in [2].

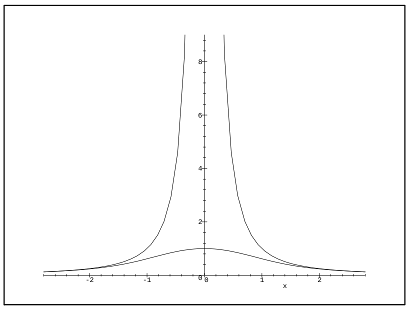

Figure 1.1. Compare Unbounded versus Bounded

Figure 1.1 compares the classical potential with a non-linear potential found in Section 6. It is also interesting to compare this with Figure 1 p.440 of [1].

Finally, we mention that a detailed analysis "at a point" in the new model [7] has led to quark-like structures.

Section 2. recalls the central dynamical equation. In Section 3. we explicitly calculate the generalized (Maxwell's) PDE's. These PDE's live on R^8, and in Section 4. we reduce them to a canonical R^4.

In Section 5. we introduce the standard electric and magnetic fields and write the general equations in terms of these.

Section 6. contains the analysis of the radially equivariant reductions of our PDE. We have some old (static) and new (non-static) examples.

We are currently investigating the rotationally symmetric case. This was discussed briefly during our presentation, but is not contained here due to lack of space.

2. Review of Martin Theory

The extension of electrodynamics that is the center of this article has its origins in the formulation of classical electrodynamics described by Souriau in [8]. This development of electrodynamics treats the vector potential as a 1-form on a space-time manifold N. Geometrically, a 1-form is a section of the cotangent bundle T^*N.

For a discussion of Souriau's work and its reformulation in the framework of geometric relativization, we refer the reader to [7]. In geometric relativization the cotangent bundle of space-time is replaced with an eight-dimensional M endowed with a neutral metric g; this space is denoted by (M^8, g).

The dynamical equations are given by

$$\iota(\Lambda)d\ell(\Lambda) - \frac{1}{2}d(\ell(\Lambda)(\Lambda)) = 0 \tag{2.1}$$

where Λ is a 2-vector field and $\ell : \Lambda_2(M) \longrightarrow \Lambda^2(M)$ is induced by the Legendre map $\ell : TM \longrightarrow T^*M$ (see Section 3.).

Remark 2.1. If Λ is simple, then 2.1 is the variation of volume formula for minimal submanifolds. Equivariant geometry and computer graphics have been used extensively to discover and investigate new examples of high dimensional minimal (and constant mean curvature) submanifolds. In this paper we investigate the non-simple Λ.

3. Balanced Fields for R^8

Let $\{e_1, \ldots, e_4, f_1, \ldots, f_4\}$ be coordinates on R^8. Define a (neutral) metric h on R^8 by $h(\frac{\partial}{\partial e_i}, \frac{\partial}{\partial f_i}) = 1$ for $1 \leq i \leq 4$ and h is 0 for all other pairs of basis vectors. Let $V, W \in so(4)$ (i.e. $V^t = -V, W^t = -W$). Let $\omega(u,v) := h(Eu, v)$ where

$$\begin{aligned} E &:= \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}, \\ &:= \begin{pmatrix} (1+WV)(1-WV)^{-1} & -2(1-WV)^{-1}W \\ 2(1-VW)^{-1}V & -(1+VW)(1-VW)^{-1} \end{pmatrix}. \end{aligned}$$

Proposition 3.1. *ω is a 2-form. In fact*

$$\begin{aligned} \omega &= 2\{(1-VW)^{-1}V\}_{ji} de_i \wedge de_j - 2\{(1-WV)^{-1}W\}_{ji} df_i \wedge df_j \\ &\quad + \{(1+WV)(1-WV)^{-1}\}_{ji} de_i \wedge df_j. \end{aligned}$$

(The sum in the first two terms is over $1 \le i < j \le 4$ and in the last term is over all $1 \le i, j \le 4$.)

Proof. Since $de_i \wedge de_j = de_i \otimes de_j - de_j \otimes de_i$, etc. we have

$$\begin{aligned} \omega(e_k, e_\ell) &= \begin{cases} \{2(1-VW)^{-1}V\}_{\ell k} & \text{if } 1 \le k < \ell \le 4 \\ 0 & \text{if } 1 \le k = \ell \le 4 \\ -\{2(1-VW)^{-1}V\}_{\ell k} & \text{if } 1 \le \ell = k \le 4, \end{cases} \\ \omega(e_k, f_\ell) &= \{(1+VW)(1-WV)^{-1}\}_{\ell k} \ \text{ if } 1 \le k, \ell \le 4, \\ \omega(f_\ell, e_k) &= -\{(1+VW)(1-WV)^{-1}\}_{\ell k} \ \text{ if } 1 \le k, \ell \le 4. \end{aligned}$$

Since $E_3 = -E_3^t$, $E_2 = -E_2^t$ and $E_4 = -E_1^t$, we see that the matrix of $\omega(u,v)$ over the basis is of the form

$$\begin{pmatrix} E_3^t & E_1^t \\ E_4^t & E_2^t \end{pmatrix}.$$

Using a Maple program one checks that this equals the matrix of $h(Eu, v)$ over the basis. □

Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_4, \mathcal{F}_1, \ldots, \mathcal{F}_4\}$ be dual to the basis $\{de_1, \ldots, de_4, df_1, \ldots, df_4\}$. Define a two-vector by

$$\begin{aligned} \Lambda &= \{2(1-VW)^{-1}V\}_{ji} \mathcal{F}_i \wedge \mathcal{F}_j - \{2(1-WV)^{-1}W\}_{ji} \mathcal{E}_i \wedge \mathcal{E}_j \\ &\quad + \{(1+WV)(1-WV)^{-1}\}_{ji} \mathcal{F}_i \wedge \mathcal{E}_j. \end{aligned}$$

The Legendre map $\ell : TM \longrightarrow T^*M$ is defined by $\ell(u)(v) := g(u,v)$. In term of a basis we have [9] $\ell(\mathcal{G}_i) = g_{ik} dg_k$. The induced map from $\Lambda_2(M)$ to $\Lambda^2(M)$ also denote by ℓ is defined by $\ell(\lambda_{ij}\mathcal{G}_i \wedge \mathcal{G}_j) := \lambda_{ij}(\ell(\mathcal{G}_i) \wedge \ell(\mathcal{G}_j))$.

Lemma 3.1. *$\ell(\Lambda) = \omega$.*

Proof. Use the definition of ℓ together with the following equations given by using the metric h:

$$\begin{aligned}
\ell(\mathcal{E}_i \wedge \mathcal{E}_j) &= \ell(\mathcal{E}_i) \wedge \ell(\mathcal{E}_j) \\
&= df_i \wedge df_j, \\
\ell(\mathcal{E}_i \wedge \mathcal{F}_j) &= df_i \wedge de_j = -de_j \wedge df_i, \\
\ell(\mathcal{F}_i \wedge \mathcal{F}_j) &= de_i \wedge de_j. \ \Box
\end{aligned}$$

Definition 3.1. *A two-vector field Λ is said to have charge e if $\mathcal{C}(\Lambda, \ell(\Lambda)) = e\, id$.*

Proposition 3.2. *Λ has charge -1.*

Proof. By Lemma 3.1 we have that $\ell(\Lambda) = \omega$. Let $\Lambda = \sum_{i<j} \Lambda_{ij}\ \mathcal{G}_i \wedge \mathcal{G}_j$, $\omega = \sum_{k<\ell} \omega_{k\ell}\ g_k \wedge g_\ell$, and $\mathcal{C}(\Lambda, \ell(\Lambda)) = \sum_{\alpha,\beta} \mathcal{C}_{\alpha\beta}\ \mathcal{G}_\alpha \wedge dg_\beta$. Using “1.1e” we have

$$\begin{aligned}
\mathcal{C}(\Lambda, \omega) &:= \sum_{i<j,k<\ell} \Lambda_{ij}\omega_{k\ell}\mathcal{C}(\mathcal{G}_i \wedge \mathcal{G}_j, dg_k \wedge dg_\ell), \\
&= \mathcal{G}_j(dg_\ell)\mathcal{G}_i \otimes dg_k - \mathcal{G}_j(dg_k)\mathcal{G}_i \otimes dg_\ell \\
&\quad -\mathcal{G}_i(dg_\ell)\mathcal{G}_j \otimes dg_k + \mathcal{G}_j(dg_k)\mathcal{G}_i \otimes dg_\ell, \\
&= \delta_{j\ell}\mathcal{G}_i \otimes dg_k - \delta_{jk}\mathcal{G}_i \otimes dg_\ell \\
&\quad -\delta_{i\ell}\mathcal{G}_j \otimes dg_k + \delta_{ik}\mathcal{G}_i \otimes dg_\ell.
\end{aligned}$$

So we have

$$\begin{aligned}
\mathcal{C}_{\alpha\beta} &= \sum_{\alpha<j,\beta<\ell} \delta_{j\ell}\Lambda_{\alpha j}\omega_{\beta\ell} - \sum_{\alpha<j,k<\beta} \delta_{jk}\Lambda_{\alpha j}\omega_{k\beta} \\
&\quad - \sum_{i<\alpha,\beta<\ell} \delta_{i\ell}\Lambda_{i\alpha}\omega_{\beta\ell} + \sum_{i<\alpha,k<\beta} \delta_{ik}\Lambda_{i\alpha}\omega_{k\beta}, \\
&= \sum_{\alpha<N,\beta<N} \Lambda_{\alpha N}\omega_{\beta N} - \sum_{\alpha<N<\beta} \Lambda_{\alpha N}\omega_{N\beta} \\
&\quad - \sum_{\beta<N<\alpha} \Lambda_{N\alpha}\omega_{\beta N} + \sum_{N<\alpha,N<\beta} \Lambda_{N\alpha}\omega_{N\beta}.
\end{aligned}$$

If we define $\tilde{\omega}$ by

$$\tilde{\omega}_{ij} = \begin{cases} \omega_{ij} & i<j \\ 0 & i=j \\ -\omega_{ij} & j<i \end{cases}$$

and $\tilde{\Lambda}$ by

$$\tilde{\Lambda}_{ij} = \begin{cases} \Lambda_{ij} & i<j \\ 0 & i=j \\ -\Lambda_{ij} & j<i, \end{cases}$$

then we have

$$\mathcal{C}_{\alpha\beta} = -\sum_{N=1}^{8} \tilde{\Lambda}_{\alpha N}\tilde{\omega}_{N\beta} = -(\tilde{\Lambda}\cdot\tilde{\omega})_{\alpha\beta}.$$

Noting that

$$\Lambda_{ij} = \begin{cases} \omega_{i+4,j+4} & 1 \le i < j \le 4 \\ -\omega_{ij} & 1 \le i \le 4, 5 \le j \le 8 \\ \omega_{i-4,j-4} & 5 \le i < j \le 8, \end{cases}$$

the result follows from a Maple program. $\square$

Lemma 3.2.

$$\begin{aligned} d\omega \;=\; & -2(\{(1-VW)^{-1}V\}_{ij,k} + \{(1-VW)^{-1}V\}_{ki,j} \\ & +\{(1-VW)^{-1}V\}_{jk,i})de_i \wedge de_j \wedge de_k \\ & +2(\{(1-WV)^{-1}W\}_{ij,k} + \{(1-WV)^{-1}W\}_{ki,j} \\ & +\{(1-WV)^{-1}W\}_{jk,i})df_i \wedge df_j \wedge df_k \\ & +2(\{(1-VW)^{-1}V\}_{ji,k} + \{(1-WV)^{-1}\}_{kj,i} \\ & -\{(1-WV)^{-1}\}_{ki,j})de_i \wedge de_j \wedge df_k \\ & +2(-\{(1-WV)^{-1}W\}_{kj,i} + \{(1-WV)^{-1}\}_{ji,k} \\ & -\{(1-WV)^{-1}\}_{ki,j})de_i \wedge df_j \wedge df_k. \end{aligned}$$

Proof. If we write

$$\omega = \sum_{1\le n<m\le 8} \omega_{nm}\, dg_n \wedge dg_m$$

for short, then

$$d\omega := \sum_{1\le n<m\le 8}\sum_{\alpha=1}^{8} \omega_{nm,\alpha}\, dg_\alpha \wedge dg_n \wedge dg_m.$$

The $de_i \wedge de_j \wedge de_k$ terms arise from exactly three terms: $\omega_{jk,i} de_i \wedge de_j \wedge de_k$, $\omega_{ik,j} de_j \wedge de_i \wedge de_k$, and $\omega_{ij,k} de_k \wedge de_i \wedge de_j$. This gives $+\omega_{jk,i} - \omega_{ik,j} + \omega_{ij,k}$ as the first coefficient. Now use $E_3^t = -E_3$. Similarly for $df_i \wedge df_j \wedge df_k$. The $de_i \wedge de_j \wedge df_k$ terms arise from exactly three terms: $\omega_{jk,i} de_i \wedge de_j \wedge df_k$, $\omega_{ik,j} de_j \wedge de_i \wedge df_k$, and $\omega_{ij,k} df_k \wedge de_i \wedge de_j$. This gives $+\omega_{jk,i} - \omega_{ik,j} + \omega_{ij,k}$ as the third coefficient. Now use $E_{1,\alpha} = \{2(1-WV)^{-1}\}_{,\alpha}$. Similarly for $de_i \wedge df_j \wedge df_k$. $\square$

Lemma 3.3.

$$dg_{i_1} \wedge \cdots \wedge dg_{i_k}(\mathcal{G}_{i_1} \wedge \cdots \wedge \mathcal{G}_{i_k}) = k!.$$

Proof. We have

$$
\begin{aligned}
LHS &= k!dg_{i_1} \wedge \cdots \wedge dg_{i_k}(\mathcal{G}_{i_1} \otimes \cdots \otimes \mathcal{G}_{i_k}) \\
&= k!k!Alt(dg_{i_1} \otimes \cdots \otimes dg_{i_k})(\mathcal{G}_{i_1} \otimes \cdots \otimes \mathcal{G}_{i_k}) \\
&= k!k!\frac{1}{k!} \sum_{\sigma \in P_r} sign(\sigma) t_\sigma (dg_{i_1} \otimes \cdots \otimes dg_{i_k})(\mathcal{G}_{i_1} \otimes \cdots \otimes \mathcal{G}_{i_k}) \\
&= k!(dg_{i_1} \otimes \cdots \otimes dg_{i_k})(\mathcal{G}_{i_1} \otimes \cdots \otimes \mathcal{G}_{i_k}) \\
&= k! \quad \square
\end{aligned}
$$

Lemma 3.4.

$$
\begin{aligned}
&\iota(\sum \lambda_{ij} \mathcal{G}_i \wedge \mathcal{G}_j)(\sum M_{k\ell n} dg_k \wedge dg_\ell \wedge dg_n) = \\
&\qquad \frac{1}{3} \sum M_{k\ell n} \lambda_{ij} dg_k \wedge dg_\ell \wedge dg_n (\mathcal{G}_i \wedge \mathcal{G}_j \wedge \mathcal{G}_\gamma) dg_\gamma .
\end{aligned}
$$

Proof. Using Lemma 3.3

$$
\begin{aligned}
&\iota(\sum \lambda_{ij} \mathcal{G}_i \wedge \mathcal{G}_j)(\sum M_{k\ell n} dg_k \wedge dg_\ell \wedge dg_n)(\mathcal{G}_\gamma) \\
&= \sum M_{k\ell n} dg_k \wedge dg_\ell \wedge dg_n (\sum \lambda_{ij} \mathcal{G}_i \wedge \mathcal{G}_j, \mathcal{G}_\gamma), \\
&= \frac{1}{3} \sum M_{k\ell n} dg_k \wedge dg_\ell \wedge dg_n ((\sum \lambda_{ij} \mathcal{G}_i \wedge \mathcal{G}_j, \mathcal{G}_\gamma) \wedge \mathcal{G}_\gamma), \\
&= \frac{1}{3} \sum M_{k\ell n} \lambda_{ij} dg_k \wedge dg_\ell \wedge dg_n (\mathcal{G}_i \wedge \mathcal{G}_j \wedge \mathcal{G}_\gamma). \quad \square
\end{aligned}
$$

Lemma 3.5.

$$\delta\omega = \iota(\Lambda) d\omega \ .$$

Proof. This is a lengthy calculation, which we omit due to lack of space. □

Proposition 3.3. *The df_k component of $\delta\omega$ is ($1 \leq i, j \leq 4$)*

$$
\begin{aligned}
&-2\{(1-VW)^{-1}V\}_{ij} \left(\{(1-WV)^{-1}W\}_{ij,k} + \{(1-WV)^{-1}W\}_{ki,j} + \{(1-WV)^{-1}W\}_{jk,i}\right) \\
&\qquad +2\{(1-WV)^{-1}W\}_{ij} \left(\{(1-VW)^{-1}V\}_{ji,k} + (1-WV)^{-1}_{kj,i} - (1-WV)^{-1}_{ki,j}\right) \\
&-2\{(1+WV)(1-WV)^{-1}\}_{ji} \left(-\{(1-WV)^{-1}W\}_{ki,j} + (1-WV)^{-1}_{ij,k} - (1-WV)^{-1}_{kj,i}\right)
\end{aligned}
$$

and the de_k component is

$$
\begin{aligned}
&-2\{(1-WV)^{-1}W\}_{ij} \left(\{(1-VW)^{-1}V\}_{ij,k} + \{(1-VW)^{-1}V\}_{ki,j} + \{(1-VW)^{-1}V\}_{jk,i}\right) \\
&\qquad +2\{(1-VW)^{-1}V\}_{ji} \left(-\{(1-WV)^{-1}W\}_{ji,k} + (1-WV)^{-1}_{ik,j} - (1-WV)^{-1}_{jk,i}\right) \\
&+2\{(1+WV)(1-WV)^{-1}\}_{ji} \left(-\{(1-VW)^{-1}V\}_{kj,i} + (1-WV)^{-1}_{ik,j} - (1-WV)^{-1}_{ij,k}\right) .
\end{aligned}
$$

Proof. Let

$$\Lambda = A_{ij}\mathcal{E}_i \wedge \mathcal{E}_j + B_{ij}\mathcal{E}_i \wedge \mathcal{F}_j + C_{ij}\mathcal{F}_i \wedge \mathcal{F}_j$$

and let

$$\begin{aligned} d\omega &= D_{ijk}de_i \wedge de_j \wedge de_k + E_{ijk}de_i \wedge de_j \wedge df_k \\ &\quad +F_{ikj}de_i \wedge df_j \wedge df_k + G_{ijk}df_i \wedge df_j \wedge df_k. \end{aligned}$$

For df_k the products EA, FB and GC occur. For de_k the products DA, EB and FC occur. Permuting, we have for df_k, the terms

$$(E_{ijk})A_{ij},$$

$$(F_{ijk} \ or \ -F_{ikj})B_{ij},$$

$$(G_{ijk} \ or \ -G_{ikj} \ or \ G_{kij})C_{ij},$$

and for de_k

$$(D_{ijk} \ or \ -D_{ikj} \ or \ D_{kij})A_ij,$$

$$(-E_{ikj} \ or \ E_{kij})B_{ij},$$

$$(F_{kij})C_{ij}.$$

It is easy to check that each "or" can be replace by "=" so only three types of indexing need occur in each sum.

Finally, we note that for α and γ symmetric with zeroes on the diagonal we have

$$\sum_{i<j}\alpha_{ij} + \sum_{i,j}\beta_{ij} + \sum_{i<j}\gamma_{ij} = \frac{1}{2}(\sum_{i,j}\alpha_{ij} + 2 * \sum_{i,j}\beta_{i,j} + \sum_{i,j}\gamma_{ij})$$

which explains the extra factor of two in the last terms.

Also we have use $A_{ij} = -A_{ji}$ and $C_{ij} = -C_{ji}$. □

In each expression there are "trace" terms. For df_k these terms are

$$\begin{aligned} &-2\{(1-VW)^{-1}V\}_{ij}\{(1-WV)^{-1}W\}_{ij,k} \\ &\quad -2\{(1-WV)^{-1}W\}_{ij}\{(1-VW)^{-1}V\}_{ij,k} \\ &\quad -2\{(1+WV)(1-WV)^{-1}\}_{ji}(1-WV)^{-1}_{ij,k} \end{aligned}$$

and for de_k these terms are

$$\begin{aligned} &-2\{(1-WV)^{-1}W\}_{ij}\{(1-VW)^{-1}V\}_{ij,k} \\ &\quad -2\{(1-VW)^{-1}V\}_{ij}\{(1-WV)^{-1}W\}_{ij,k} \\ &\quad -2\{(1+WV)(1-WV)^{-1}\}_{ji}(1-WV)^{-1}_{ij,k}. \end{aligned}$$

Note that these expressions are identical. We work with the second one. Using the product rule on the first term and using $(1+WV)(1-WV)^{-1} = (1-WV)^{-1}(1+WV)$ on the third term we have

$$
\begin{aligned}
-2[&\{(1-WV)^{-1}W\}_{ij}\{(1-VW)^{-1}V\}_{ij}]_{,k}\\
&+2\{(1-WV)^{-1}W\}_{ij,k}\{(1-VW)^{-1}V\}_{ij}\\
&-2\{(1-VW)^{-1}V\}_{ij}\{(1-WV)^{-1}W\}_{ij,k}\\
&-2\{(1-WV)^{-1}WV\}_{ij}(1-WV)^{-1}_{ji,k}-2(1-WV)^{-1}_{ij}(1-WV)^{-1}_{ji,k}.
\end{aligned}
$$

Note that the second and third terms cancel (commute the numbers computed for each i,j,k). Using the product rule on the fourth term (and noting that $\{(1-WV)^{-1}WV\}_{ij,k}=(1-WV)^{-1}_{ij,k}$) we obtain

$$
\begin{aligned}
-2[&\{(1-WV)^{-1}W\}_{ij}\{(1-VW)^{-1}V\}_{ij}]_{,k}\\
&-2[\{(1-WV)^{-1}WV\}_{ij}(1-WV)^{-1}_{ji}]_{,k}+2(1-WV)^{-1}_{ij,k}(1-WV)^{-1}_{ji}\\
&-2(1-WV)^{-1}_{ij}(1-WV)^{-1}_{ji,k}.
\end{aligned}
$$

Transposing the second product of the first term yields a change in sign so that, considered as traces, the first term then cancels the second term. The third and fourth terms cancel. Hence the "trace" terms all add to zero.

Next, consider the "divergence" terms in k. For df_k they are (again we have use $A_{ij}=-A_{ji}$ and $C_{ij}=-C_{ji}$ and also $(1+WV)(1-WV)^{-1}=((1+VW)(1-VW)^{-1})^t$):

$$
+4\{(1-WV)^{-1}W\}_{ij}(1-WV)^{-1}_{kj,i}+2\{(1+WV)(1-WV)^{-1}\}_{ji}\{(1-WV)^{-1}W\}_{ki,j} \tag{3.1}
$$

and

$$
-4\{(1-VW)^{-1}V\}_{ij}\{(1-WV)^{-1}W\}_{ki,j}+2\{(1+WV)(1-WV)^{-1}\}_{ji}(1-WV)^{-1}_{kj,i} \tag{3.2}
$$

For de_k they are

$$
+4\{(1-VW)^{-1}V\}_{ji}(1-WV)^{-1}_{ik,j}+2\{(1+VW)(1-VW)^{-1}\}_{ji}\{(1-VW)^{-1}V\}_{ki,j} \tag{3.3}
$$

and

$$
-4\{(1-WV)^{-1}W\}_{ij}(1-VW)_{ik,j}+2\{(1+VW)(1-VW)^{-1}\}_{ji}(1-VW)^{-1}_{kj,i} \tag{3.4}
$$

Noting that $\{(1-VW)^{-1}\}^t=(1-WV)^{-1}$ one sees that these expressions can be transformed into one another by $V\longleftrightarrow W$. First consider (3.1). Using $(1-WV)^{-1}(1+WV)=(1+WV)(1-WV)^{-1}$ and changing notation slightly we obtain

$$
\begin{aligned}
= \quad & 4\{(1-WV)^{-1}W\}_{ij}(1-WV)^{-1}_{kj,i}\\
& +2\{(1-WV)^{-1}(1+WV)\}_{ij}\{(1-WV)^{-1}W\}_{kj,i}
\end{aligned}
$$

Again using $\{(1-VW)^{-1}\}^t=(1-WV)^{-1}$ on the first term we have

$$
\begin{aligned}
= \quad & 4\{(1-WV)^{-1}W\}_{ij}(1-VW)^{-1}_{jk,i}\\
& +2\{(1-WV)^{-1}(1+WV)\}_{ij}\{(1-WV)^{-1}W\}_{kj,i}
\end{aligned}
$$

Spliting the first term using the product rule, breaking the second term into two parts and slightly changing notation we obtain

$$\begin{aligned}
= \quad & 4(1-WV)^{-1}_{ij}\{W(1-VW)^{-1}\}_{jk,i} \\
& -4(1-WV)^{-1}_{i\ell}W_{\ell j,i}(1-VW)^{-1}_{jk} \\
& +2(1-WV)^{-1}_{ij}\{(1-WV)^{-1}W\}_{kj,i} \\
& +2\{(1-WV)^{-1}WV\}_{ij}\{(1-WV)^{-1}W\}_{kj,i}
\end{aligned}$$

Using $W(1-VW)^{-1} = -\{(1-WV)^{-1}W\}^t$ the first and third terms add. Their sum combines nicely with the fourth term, and we have

$$= \quad 2\{(1-WV)^{-1}W\}_{ik,i} - 4(1-WV)^{-1}_{i\ell}W_{\ell j,i}(1-VW)^{-1}_{jk}.$$

Next consider 3.4, which using $(1-WV)^{-1}(1+WV) = (1+WV)(1-WV)^{-1}$ and changing notation slightly becomes

$$\begin{aligned}
= \quad & -4\{(1-WV)^{-1}W\}_{ij}\{(1-VW)^{-1}V\}_{ki,j} \\
& +2\{(1-WV)^{-1}(1+WV)\}_{ij}(1-WV)^{-1}_{jk,i}.
\end{aligned}$$

Using $(1-VW)^{-1}V = -\{V(1-WV)^{-1}\}^t$ on the first term, the product rule on the second term and changing notation slightly we obtain

$$\begin{aligned}
= \quad & -4\{(1-WV)^{-1}W\}_{ji}\{V(1-WV)^{-1}\}_{ik,j} \\
& +2(1-WV)^{-1}_{ij}\{(1+WV)(1-WV)^{-1}\}_{jk,i} \\
& -2(1-WV)^{-1}_{i\ell}(1+WV)_{\ell j,i}(1-WV)^{-1}_{jk}.
\end{aligned}$$

Using the product rule on the first term, expanding the second term and changing notation slightly we obtain

$$\begin{aligned}
= \quad & -4(1-WV)^{-1}_{ji}\{WV(1-WV)^{-1}\}_{ik,j} + 4(1-WV)^{-1}_{j\ell}W_{\ell i,j}\{V(1-WV)^{-1}\}_{ik} \\
& +2(1-WV)^{-1}_{ji}\{WV(1-WV)^{-1}\}_{ik,j} + 2(1-WV)^{-1}_{ji}(1-WV)^{-1}_{ik,j} \\
& -2(1-WV)^{-1}_{i\ell}(WV)_{\ell j,i}(1-WV)^{-1}_{jk}.
\end{aligned}$$

Applying the product rule to the sum of the first and third terms, and changing notation slightly we obtain

$$\begin{aligned}
= \quad & -2(1-WV)^{-1}_{i\ell}(WV)_{\ell j}(1-WV)^{1}_{jk,i} \\
& +2(1-WV)^{-1}_{ij}(1-WV)^{-1}_{jk,i} \\
& -2(1-WV)^{-1}_{i\ell}(WV)_{\ell j,i}(1-WV)^{-1}_{jk} \\
& +4(1-WV)^{-1}_{j\ell}W_{\ell i,j}V_{im}(1-WV)^{-1}_{mk} \\
& -2(1-WV)^{-1}_{i\ell}(WV)_{\ell j,i}(1-WV)^{-1}_{jk}.
\end{aligned}$$

The sum of the first and second terms simplifies nicely. We apply the product rule to the fourth term and add the third and fifth terms. Changing notation slightly we obtain

$$\begin{aligned} &= 2(1-WV)^{-1}_{ik,i} \\ &\quad +4(1-WV)^{-1}_{j\ell}(WV)_{\ell m,j}(1-WV)^{-1}_{mk} \\ &\quad -4(1-WV)^{-1}_{j\ell}W_{\ell i}V_{im,j}(1-WV)^{-1}_{mk} \\ &\quad -4(1-WV)^{-1}_{i\ell}(WV)_{\ell j,i}(1-WV)^{-1}_{jk}. \end{aligned}$$

The second and fourth terms add to zero. Changing notation slightly we obtain

$$= 2(1-WV)^{-1}_{jk,j} - 4\{(1-WV)^{-1}W\}_{ji}V_{im,j}(1-WV)^{-1}_{mk}.$$

Consequently,

$$\begin{aligned} \delta\omega &= \sum_k [2\{(1-WV)^{-1}W\}_{ik,i} - 4(1-WV)^{-1}_{i\ell}W_{\ell j,i}(1-VW)^{-1}_{jk} \\ &\quad +2(1-VW)^{-1}_{jk,j} - 4\{(1-VW)^{-1}V\}_{ji}W_{im,j}(1-VW)^{-1}_{mk}]df_k \\ &+ \sum_k [2\{(1-VW)^{-1}V\}_{ik,i} - 4(1-VW)^{-1}_{i\ell}V_{\ell j,i}(1-WV)^{-1}_{jk} \\ &\quad +2(1-WV)^{-1}_{jk,j} - 4\{(1-WV)^{-1}W\}_{ji}V_{im,j}(1-WV)^{-1}_{mk}]de_k. \end{aligned}$$

4. The First Reduction - to a Euclidean R^4

We are now going to change coordinates. The reason we didn't start with the new x, y coordinates is that the previous calculations would have been been even worse (involving square roots). Let $x_i = \frac{1}{2}(e_i + f_i)$ and $y_i = \frac{1}{2}(e_i - f_i)$. Suppose that V and W depend only on $(x_1, \ldots, x_4)$. Then (temporarily using superscripts ef and xy)

$$\begin{aligned} V^{ef}_{ij}(e_1, \ldots, e_4, f_1, \ldots, f_4) &= V^{xy}_{ij}(\frac{e_1+f_1}{2}, \ldots, \frac{e_4+f_4}{2}), \\ W^{ef}_{ij}(e_1, \ldots, e_4, f_1, \ldots, f_4) &= W^{xy}_{ij}(\frac{e_1+f_1}{2}, \ldots, \frac{e_4+f_4}{2}). \end{aligned}$$

Consequently, $V^{ef}_{ij,k} = \frac{1}{2}V^{xy}_{ij,k}$ and $W^{ef}_{ij,k} = \frac{1}{2}W^{xy}_{ij,k}$. Using these identities and droping the superscripts, our equation for $\delta\omega$ only changes by a factor of two. We have

$$\begin{aligned} \delta\omega &= \sum_k [\{(1-WV)^{-1}W\}_{ik,i} - 2(1-WV)^{-1}_{i\ell}W_{\ell j,i}(1-VW)^{-1}_{jk} \\ &\quad +(1-VW)^{-1}_{jk,j} - 2\{(1-VW)^{-1}V\}_{ji}W_{im,j}(1-VW)^{-1}_{mk}]df_k \\ &+ \sum_k [\{(1-VW)^{-1}V\}_{ik,i} - 2(1-VW)^{-1}_{i\ell}V_{\ell j,i}(1-WV)^{-1}_{jk} \\ &\quad +(1-WV)^{-1}_{jk,j} - 2\{(1-WV)^{-1}W\}_{ji}V_{im,j}(1-WV)^{-1}_{mk}]de_k. \end{aligned}$$

We can simplify the df_k component of $\delta\omega$. Reindexing, using $(1-WV)^{-1}W = W(1-VW)^{-1}$ and applying the product rule to the second and fourth terms we have

$$\begin{aligned} = \;& \{(1-WV)^{-1}W\}_{ik,i} - 2(1-WV)^{-1}_{i\ell}\{(1-WV)^{-1}W\}_{\ell k,i} \\ & +2\{(1-WV)^{-1}W\}_{i\ell}(1-VW)^{-1}_{\ell k,i} \\ & +(1-VW)^{-1}_{ik,i} - 2\{(1-VW)^{-1}V\}_{ij}\{(1-WV)^{-1}W\}_{jk,i} \\ & +2\{(1-VW)^{-1}(1-1+VW)\}_{ij}(1-VW)^{-1}_{jk,i}. \end{aligned}$$

Combining the first, second and fifth terms and (carefully) combining the third, fourth and sixth terms we have

$$\begin{aligned} = \;& \{1-2(1-WV)^{-1} - 2(1-VW)^{-1}V\}_{ij}\{(1-WV)^{-1}W\}_{jk,i} \\ & -\{1-2(1-VW)^{-1} - 2(1-WV)^{-1}W\}_{ij}(1-VW)^{-1}_{jk,i}. \end{aligned}$$

A similar expression holds for the de_k component obtained by exchanging V and W. Hence, we arrive at

$$\begin{aligned} \delta\omega = & \sum_k (1-2(1-WV)^{-1}-2(1-VW)^{-1}V)_{ij}(((1-WV)^{-1}W)_{jk,i}df_k+(1-WV)^{-1}_{jk,i}de_k) \\ & -\sum_k (1-2(1-VW)^{-1}-2(1-WV)^{-1}W)_{ij}(((1-VW)^{-1}V)_{jk,i}de_k+(1-VW)^{-1}_{jk,i}df_k). \end{aligned} \tag{4.1}$$

5. Electric and Magnectic Fields

We now introduce notation which resembles the classical notation.

$$V = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \tag{5.1}$$

$$W = \begin{pmatrix} 0 & F_1 & F_2 & F_3 \\ -F_1 & 0 & C_3 & -C_2 \\ -F_2 & -C_3 & 0 & C_1 \\ -F_3 & C_2 & -C_1 & 0 \end{pmatrix}. \tag{5.2}$$

For short (introducing some new notation), we have

$$V = \begin{pmatrix} 0 & E \\ -E & *B \end{pmatrix}, \tag{5.3}$$

$$W = \begin{pmatrix} 0 & F \\ -F & *C \end{pmatrix}. \tag{5.4}$$

Also, we define

$$\tilde{V} = \begin{pmatrix} 0 & B \\ -B & -E \end{pmatrix}, \tag{5.5}$$

$$\tilde{W} = \begin{pmatrix} 0 & C \\ C & -F \end{pmatrix}. \tag{5.6}$$

The following formulas are obtained via matrix calculations and are checked in a Maple program.

$$VW = \begin{pmatrix} -E \cdot F & -E \times C \\ B \times F & -(B \cdot C)\, I + E \otimes F + C \otimes B \end{pmatrix}, \tag{5.7}$$

where

$$E \otimes F = \begin{pmatrix} E_1F_1 & E_1F_2 & E_1F_3 \\ E_2F_1 & E_2F_2 & E_2F_3 \\ E_3F_1 & E_3F_2 & E_3F_3 \end{pmatrix}. \tag{5.8}$$

$$VWV = \alpha V + \beta \tilde{W},$$
$$WVW = \alpha W + \delta \tilde{V},$$

where $\alpha = -E \cdot F - B \cdot C$, $\beta = B \cdot E$ and $\delta = C \cdot F$.

$$\tilde{V}V = V\tilde{V} = -\beta I,$$
$$\tilde{W}W = W\tilde{W} = -\delta I,$$
$$VWVW = \alpha VW - \beta\delta I,$$
$$WVWV = \alpha WV - \beta\delta I.$$

We obtain finally

$$\begin{aligned}
(1 - WV)^{-1} &= \frac{1-\alpha}{1-\alpha+\beta\delta} I + \frac{1}{1-\alpha+\beta\delta} WV, \\
(1 - VW)^{-1} &= \frac{1-\alpha}{1-\alpha+\beta\delta} I + \frac{1}{1-\alpha+\beta\delta} VW, \\
(1 - WV)^{-1}W &= \frac{1}{1-\alpha+\beta\delta} W + \frac{\delta}{1-\alpha+\beta\delta} \tilde{V}, \\
(1 - VW)^{-1}V &= \frac{1}{1-\alpha+\beta\delta} V + \frac{\beta}{1-\alpha+\beta\delta} \tilde{W}.
\end{aligned}$$

By introducing the functions

$$q = \frac{1-\alpha}{1-\alpha+\beta\delta}, h = \frac{1}{1-\alpha+\beta\delta}, g = \frac{\beta}{1-\alpha+\beta\delta}, k = \frac{\delta}{1-\alpha+\beta\delta},$$

these equations become

$$\begin{aligned}
(1 - WV)^{-1} &= qI + hWV, \\
(1 - VW)^{-1} &= qI + hVW, \\
(1 - WV)^{-1}W &= hW + k\tilde{V}, \\
(1 - VW)^{-1}V &= hV + g\tilde{W}.
\end{aligned} \tag{5.9}$$

6. Radially Symmetric

We now consider, within the R^4 reduction of the last section, both the static and the time-varying radially symmetric cases.

Radial symmetry and time independence implies that V and W can be expressed in terms of four functions $e(r), b(r), f(r), c(r)$ where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and have the form

$$V = \begin{pmatrix} 0 & ex \\ -ex & b(*x) \end{pmatrix}, \quad W = \begin{pmatrix} 0 & fx \\ -fx & f(*x) \end{pmatrix},$$

where $x = (x_1, x_2, x_3)$. Using 5.9 and 4.1, we calculate $d\omega$ in Maple programs.

In the time dependent case we have

$$\begin{aligned}
((\dot{e}f - \dot{f}e) - (e'f^2r^2 + f'))r(bcr^2+1) + f(fer^2+1)(bcr^2-3) &= 0,\\
((f^2r^2\dot{e} + \dot{f}) + (e'f - f'e)r^2)(bcr^2+1) + 2r(bc+ef)(1+efr^2) &= 0,\\
((\dot{f}e - \dot{e}f) - (f'e^2r^2 + e'))r(bcr^2+1) + e(fer^2+1)(bcr^2-3) &= 0,\\
((e^2r^2\dot{f} + \dot{e}) + (f'e - e'f)r^2)(bcr^2+1) + 2r(bc+ef)(1+efr^2) &= 0,
\end{aligned}$$

and in the static case

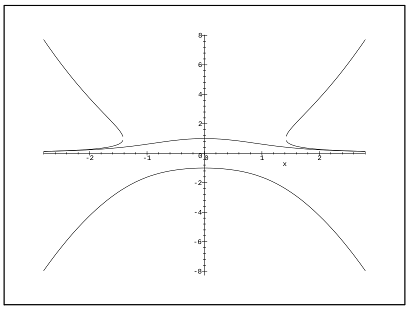

Figure 6.1. Static Solutions

$$\begin{aligned}
(e'f^2r^2 + f')r(bcr^2+1) + f(fer^2+1)(bcr^2-3) &= 0,\\
(e'f - f'e)r^2(bcr^2+1) + 2r(bc+ef)(1+efr^2) &= 0,\\
(f'e^2r^2 + e')r(bcr^2+1) + e(fer^2+1)(bcr^2-3) &= 0,\\
(f'e - e'f)r^2(bcr^2+1) + 2r(bc+ef)(1+efr^2) &= 0.
\end{aligned}$$

In the static case we can solve uniquely. The equations lead to $\rightsquigarrow bc = -ef$, $\rightsquigarrow e = kf$, $\rightsquigarrow bc = -k^2f^2$

$$\rightsquigarrow f'r(-k^2f^2r^2+1)+f(k^2f^2r^2+3)=0.$$

The solution is (see Figure 6.1)

$$f(r)=\frac{r^2\pm\sqrt{r^4-4kC^4}}{2rkC^2}.$$

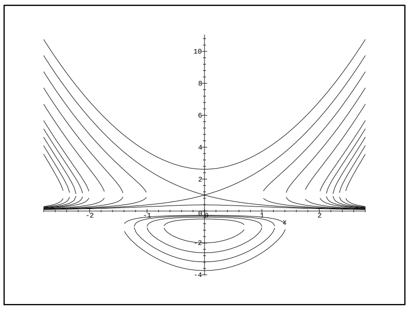

Figure 6.2. The Time=0 Slice of a Family of Non-Static Solutions

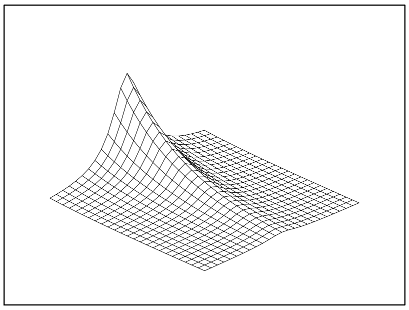

Figure 6.3. A Typical Non-Static Solution

In the time-varying case we were able to find the following solution (see Figures 6.2 and 6.3).

$$\begin{aligned} e &= f, \\ bcr^2 &= \frac{-fr^2+frc_1}{1-frc}, \\ fr &= e^{-2c_1t}(c_1-\frac{r^2c_2}{2}\pm\frac{1}{2}\sqrt{4c_1^2-4c_1^2r^2c_2+r^4c_2^2-4}). \end{aligned}$$

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