# Finite Type Lorentz Harmonic Maps and the Method of Symes

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**Abstract**

The purpose of this note is to define the notion of Symes type for Lorentz harmonic maps into $S^2(1)$ and to prove that Lorentz harmonic maps into $S^2(1)$ are finite-Symes type if and only if they are finite type. The analogous theorem for harmonic maps is a special case of a theorem due to Burstall and Pedit which states that harmonic maps into k-symmetric spaces are finite-Symes type if and only if they are finite type.

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# 1 Introduction

The purpose of this note is to define the notion of Symes type for Lorentz harmonic maps into $S^2(1)$ and to prove that Lorentz harmonic maps into $S^2(1)$ are finite-Symes type if and only if they are finite type. The analogous theorem for harmonic maps is a special case of a theorem due to Burstall and Pedit [BP] which states that harmonic maps into k-symmetric spaces are finite-Symes type if and only if they are finite type. The term "finite-Symes type" is ours, but is very close to the terminology used in [BP].

In both cases (harmonic and Lorentz harmonic) the story line is related to the DPW representation of harmonic maps [DPW]. A DPW-like representation for (weakly regular) Lorentz harmonic maps into $S^2(1)$ was given by Toda [T] in her study of (weakly regular) pseudo-spherical surfaces. In particular this includes all Lorentz harmonic maps into $S^2(1)$ of finite type (or, as we will show, finite-Symes type).

It is important to note that Toda's DPW-like representation produces all (weakly regular) Lorentz harmonic maps, including many which are not finite type. If one restricts to finite type maps, then an excellent reference is Terng-Uhlenbeck's far-reaching work [TU]. They do not consider the question addressed in this paper. For another source of Burstall and Pedit's result, and many related topics, see Guest's book [G]. Guest introduces a notion of "generalized finite type", it would be interesting to know if any of these are not finite type.

See [MS1] or [T] for the relationship between Lorentz harmonic maps in $S^2(1)$, pseudo-spherical surfaces in $R^3$ (K-surfaces) and the sine-Gordon equation. See [PS] for the historical and geometrical meaning of "finite type" in the context of soliton geometry and harmonic maps. In the physic literature the term "finite gap" is used.

For completeness and comparison we excised the harmonic case from [BP] and include it as the first half of the paper. Our proof follows the pattern of this paper. But due to the double loop group setting required for K-surfaces our proof has some new features. We would expect that at some time there will be a more general theory including most, if not all, present DPW-like theories. Then there also should be a general theorem relating finite type and finite-Symes type. We would be pleased if this note would be a steping stone towards such a general result. A preliminary attempt at this was already done in [MS2].

One central theme of this paper is to highlight the similarities and differences between the harmonic and the Lorentz harmonic cases. In order to better see the big picture. Particularly interesting is the delicate interplay between the Iwasawa and the Birkhoff factorization theorems. There has been recent work on this by I. McIntosh [S2].

In general in order to construct Lorentz harmonic maps one must first solve a (loop) system of ODE's and then carry out a Birkhoff factorization. For finite type solutions the factorization step can be skipped. For finite-Symes type the solving of the ODE's step can be skipped. It is the result of this paper that these apparently very different cheating tricks produce exactly the same set of solutions.

Finite type Lorentz harmonic maps enjoy many properties not shared by non-finite type Lorentz harmonic maps. Here we mention just a few.

- They can be constructed by solving a ODE's [MS1].
- Alternatively, due to our main result (Theorem 3.2), they can be constructed by a Birkhoff factorization.
- They can be constructed by theta functions [D], [M]. (Even in the case of a singular spectral curve [E].)
- They can be classified via their spectral curves [KK].
- They can be deformed isospectrally via linear flows on the Jacobian of their spectral curves [EF].
- They include the Gauss maps of many of the classical K-surfaces, such as the pseudo-sphere, and the surfaces of Dini, Minding, Kuen, Enneper, and Dobriner. See [MS2] for a history of the subject and for several examples which are not finite type.

In another paper by Burstall and Pedit [BP2] they prove that every finite type harmonic map is in the dressing orbit of the trivial map. It is not known if this is also true for finite type Lorentz harmonic maps.

The tools developed in this paper are currently being implemented numerically and it is hoped that soon a framework for studying K-surfaces can be created as Nick Schmidt [S1] has done for H-surfaces (surfaces with constant mean curvature).

Finally, similar results might hold for Hasimoto surfaces and the Nonlinear Schroedinger Equation. The loop group approach for these examples is less developed.

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# 2 Harmonic Maps into a Symmetric Space

## 2.1 Loop Spaces

Let $N = G/K$ be a compact Riemannian symmetric space with $G$ a compact semisimple Lie group, $\pi$ the coset projection, $K$ the fixed set of an order two automorphism $\tau$. Assume the corresponding Cartan decomposition is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}.$$

An element $v$ in $\mathfrak{g}$ decomposes as $v_{\mathfrak{k}} + v_{\mathfrak{m}}$. We consider the Iwasawa decomposition of $\mathfrak{k}^{\mathbb{C}}$ in three forms. First

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

where $\mathfrak{a} := i\mathfrak{t}$ for some maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and $\mathfrak{n}$ is the nilpotent subalgebra given by the positive root spaces. Secondly, if we let $\mathfrak{b} := \mathfrak{a} \oplus \mathfrak{n}$ (a solvable subalgebra), then we have

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{b}.$$

Thirdly, if we let

$$\mathfrak{h} := \mathfrak{t}^{\mathbb{C}} = \mathfrak{t} \oplus \mathfrak{a}$$

then

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{n} \oplus \mathfrak{h} \oplus \bar{\mathfrak{n}}.$$

The third form allows us to define a map $r : \mathfrak{k}^{\mathbb{C}} \to \mathfrak{k}^{\mathbb{C}}$ by

$$r(\eta) := \eta_{\bar{\mathfrak{n}}} + \frac{1}{2}\eta_{\mathfrak{h}}, \tag{2.1}$$

where the subscript denotes the component of $\eta$ in the corresponding subspace.

The second form is used in the definition of (twisted) loop algebras. Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be the involution determining $N$ (and $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ its extension).

Choose and fix $s > \frac{1}{2}$ and in the following definitions consider only maps which are (restrictions of) maps in $H^s$.

$$\begin{aligned}
\Lambda\mathfrak{g}^{\mathbb{C}}_{\tau} &:= \{\xi : S^1 \to \mathfrak{g}^{\mathbb{C}} : \xi(-\lambda) = \tau\xi(\lambda)\ \forall\lambda \in S^1\};\\
\Lambda\mathfrak{g}_{\tau} &:= \{\xi \in \Lambda\mathfrak{g}^{\mathbb{C}}_{\tau} : \xi : S^1 \to \mathfrak{g}\};\\
\Lambda_+\mathfrak{g}^{\mathbb{C}}_{\tau} &:= \{\xi \in \Lambda\mathfrak{g}^{\mathbb{C}}_{\tau} : \xi \text{ extends holomorphically to } \xi : D \to \mathfrak{g}^{\mathbb{C}} \text{ and } \xi(0) \in \mathfrak{b}\};
\end{aligned}$$

where $D$ is the disc $\{|\lambda| < 1\}$.

By considering the Fourier expansions we have (bewaring of reverse indexing)

$$\begin{aligned}
\Lambda\mathfrak{g}^{\mathbb{C}}_{\tau} &= \{\xi = \sum_{n\in\mathbb{Z}} \xi_{-n}\lambda^n : \xi_n \in \mathfrak{g}^{\mathbb{C}} \text{ and } \tau\xi_n = (-1)^n\xi_n\ \forall n\};\\
\Lambda\mathfrak{g}_{\tau} &= \{\xi \in \Lambda\mathfrak{g}^{\mathbb{C}}_{\tau} : \overline{\xi_n} = \xi_{-n}\ \forall n\};\\
\Lambda_+\mathfrak{g}^{\mathbb{C}}_{\tau} &= \{\xi \in \Lambda\mathfrak{g}^{\mathbb{C}}_{\tau} : \xi_n = 0 \text{ for } n > 0 \text{ and } \xi_0 \in \mathfrak{b}\}.
\end{aligned}$$

To introduce the infinite-dimensional (loop) Lie groups that correspond to these (loop) Lie algebras we let $K^{\mathbb{C}} =: KB$ be the Iwasawa decomposition of $K^{\mathbb{C}}$ corresponding to that of $\mathfrak{k}^{\mathbb{C}}$ (using the second form). Then we have:

$$\begin{aligned}
\Lambda G^{\mathbb{C}}_{\tau} &= \{\gamma : S^1 \to G^{\mathbb{C}} : \gamma(-\lambda) = \tau\gamma(\lambda)\ \forall\ \lambda \in S^1\};\\
\Lambda G_{\tau} &= \{\gamma \in \Lambda G^{\mathbb{C}}_{\tau} : \gamma : S^1 \to G\};\\
\Lambda_+ G^{\mathbb{C}}_{\tau} &= \{\gamma \in \Lambda G^{\mathbb{C}}_{\tau} : \gamma \text{ extends holomorphically to } \gamma : D \to G^{\mathbb{C}}, \gamma(0) \in B\}.
\end{aligned}$$

At the loop algebra level the (twisted) Iwasawa vector space decomposition is given by :

$$\Lambda\mathfrak{g}^{C}_{\tau} = \Lambda\mathfrak{g}_{\tau} \oplus \Lambda_+\mathfrak{g}^{\mathbb{C}}_{\tau};$$

and at the loop group level the (twisted) Iwasawa decomposition is given by an onto diffeomorphism (via multiplication) [DPW]:

$$\Lambda G_{\tau} \times \Lambda_+ G^{\mathbb{C}}_{\tau} \to \Lambda G^{\mathbb{C}}_{\tau}.$$

We will denote the decomposition of an element $\gamma \in \Lambda G^{\mathbb{C}}_{\tau}$ via

$$\gamma = \gamma_u\gamma_p.$$

Finally, let

$$\Lambda_d := \{\xi \in \Lambda\mathfrak{g}_{\tau} : \xi_n = 0 \text{ for all } |n| > d\}.$$

## 2.2 Maps

Let $M$ be a Riemann surface and $\phi : M \to N$. A framing of $\phi$ is a map $\Phi : M \to G$ such that $\pi \circ \Phi = \phi$. Such lifts always exist locally. Let $\Phi$ be a framing of $\phi$, $\alpha := \Phi^{-1} d\Phi$ and $(\lambda \in S^1)$

$$\alpha^\lambda := \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{m}}, \tag{2.2}$$

where $\alpha_{\mathfrak{m}}$ decomposes into its $(1,0)$ and $(0,1)$-parts $\alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$. A standard result is that if $\phi$ is harmonic then $\alpha^\lambda$ satisfies the Zero Curvature Equation (ZCE)

$$d\alpha^\lambda + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0.$$

Conversely, if $\alpha^\lambda$ is a loop of 1-forms of the form (2.2), then locally $d\Phi^\lambda = \Phi^\lambda\alpha^\lambda$ can be (smoothly) integrated and projected to yield a loop of harmonic maps to $N$.

A map $a : M \to \Lambda G_\tau$ is called an extended framing if

$$a^{-1}da = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_0 + \lambda\alpha''_{\mathfrak{m}}.$$

Note that any extended framing gives rise to a loop of harmonic maps (since $\alpha^\lambda := a^{-1}da$ satisfies the ZCE). Also note again that any harmonic map on a simply connected $M$ defines an extended framing (since $da = a\alpha^\lambda$ can be smoothly integrated).

Let $d \equiv 1 \bmod 2$. If $\xi : \mathbb{R}^2 \to \Lambda_d$ then $\alpha_\xi$, a $\Lambda\mathfrak{g}_\tau$-valued 1-form on $\mathbb{R}^2$, is defined by

$$\alpha_\xi := (\lambda^{-1}\xi_d + r(\xi_{d-1}))dz + (\lambda\xi_{-d} + \overline{r(\xi_{d-1})})d\bar{z}, \tag{2.3}$$

where $r$ is defined in (2.1). Note that $d-1$ is even and hence $\xi_{d-1}$ is in $\mathfrak{k}^{\mathbb{C}}$.

It is pivotal to the theory that such an $\alpha_\xi$ automatically satisfies the ZCE as soon as $\xi$ satisfies $d\xi = [\xi, \alpha_\xi]$ (see, for example, [BP]).

## 2.3 Main Result

A map $\Phi^\lambda : \mathbb{R}^2 \longrightarrow \Lambda G_\tau$ is called finite type if $d\Phi^\lambda = \Phi^\lambda\alpha_\xi$ , $\Phi^\lambda(0) = I$ where $\xi : \mathbb{R}^2 \to \Lambda_d$ and $d\xi = [\xi, \alpha_\xi]$ , $\xi(0) = \xi_o$ for some $\xi_o \in \Lambda_d$ where $d \equiv 1 \bmod 2$. The loop $\xi$ is called the Killing field of $\Phi^\lambda$. It is a well-known result that every $\xi_o \in \Lambda_d$ with $d \equiv 1 \bmod 2$ yields such a map of finite type (again see, for example, [BP]). The function $\det(\xi(\lambda) - \mu I)$ is called the spectral curve of $\Phi^\lambda$ and does not depend on $z$. A harmonic map $\phi : \mathbb{R}^2 \longrightarrow G/K$ is called finite type if $\phi = \pi \circ ev_1 \circ \Phi^\lambda$ where $\Phi^\lambda : \mathbb{R}^2 \longrightarrow \Lambda G_\tau$ is finite type. (Here $ev_1$ means evaluation at $\lambda = 1$.)

A map $a : \mathbb{R}^2 \longrightarrow \Lambda G_\tau$ is called finite-Symes type if $a = \exp(z\lambda^{d-1}\xi_o)_u$ for some $\xi_o \in \Lambda_d$ where $d \equiv 1 \bmod 2$. Note that every $\xi_o \in \Lambda_d$ with $d \equiv 1 \bmod 2$ yields a map of finite-Symes type via the Iwasawa decomposition of $\exp(z\lambda^{d-1}\xi_o)$. A harmonic map $\phi : \mathbb{R}^2 \longrightarrow G/K$ is called finite-Symes type if $\phi = \pi \circ ev_1 \circ a$ where $a : \mathbb{R}^2 \longrightarrow \Lambda G_\tau$ is finite-Symes type.

We remark that more generally a map $a : \mathbb{R}^2 \longrightarrow \Lambda G_\tau$ is called Symes type if $a = \exp(z\eta_o)_u$ for some $\eta_o \in \Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ such that $\lambda\eta_o$ extends holomorphically to $D$.

In [BP] it is shown that if $a$ is of Symes type than it is an extending framing. In particular it is shown there (p. 258) that

$$a^{-1}da = \lambda^{-1}\eta_1 dz + (\eta_0 dz)_{\mathfrak{k}} + \lambda\bar{\eta}_1 d\bar{z} \tag{2.4}$$

where $\eta := \mathrm{Ad}\, a^{-1}\eta_o$ and $\eta =: \sum_{n\geq -1} \lambda^n \eta_{-n}$.

**Theorem 2.1.** **[BP]** *A harmonic map $\phi : \mathbb{R}^2 \longrightarrow G/K$ is finite type if and only if it is finite-Symes type.*

**Proof:** ($\Leftarrow$) First we will show that if $\phi$ is finite-Symes type, then it is finite type. If it is finite-Symes type, then, by definition, $\phi = \pi \circ ev_1 \circ a$, where $a$ is finite-Symes type. That is, $a = \exp(z\lambda^{d-1}\xi_o)_u$, as above, for some $\xi_o \in \Lambda_d$. Let $\Phi^\lambda := a$ and let $\xi := \Phi^{\lambda^{-1}}\xi_o\Phi^\lambda$. We show $\Phi^\lambda$ is finite type (and hence $\phi$ is), by showing that $\xi$ is the desired Killing field for $\Phi^\lambda$. That is, $d\xi = [\xi, \alpha_\xi]$, where $\alpha_\xi$ is given by 2.3. Furthermore, by definition, we require that $\xi : \mathbb{R}^2 \to \Lambda_d$ and $\xi(0) = \xi_o$. In summary we need to show that

i) $\xi : \mathbb{R}^2 \longrightarrow \Lambda_d$,

ii) $\Phi^{\lambda^{-1}} d\Phi^\lambda = (\lambda^{-1}\xi_d + r(\xi_{d-1}))dz + (\lambda\xi_{-d} + \overline{r(\xi_{d-1})})d\bar{z}$,

iii) $d\xi = [\xi, \Phi^{\lambda^{-1}} d\Phi^\lambda]$ with $\xi(0) = \xi_o$.

Let $b$ denote the "positive" factor of the Iwasawa decomposition of $\exp(z\lambda^{d-1}\xi_o)$. That is, $b = \Phi^{\lambda^{-1}} \exp(z\lambda^{d-1}\xi_o)$. For (i) first note $\xi = \mathrm{Ad}(b)\xi_o$ since

$$b\xi_o b^{-1} = \Phi^{\lambda^{-1}} \mathrm{Ad}(\exp(z\lambda^{d-1}\xi_o))\xi_o\Phi = \xi.$$

Then note $\xi \in \Lambda_d$ if and only if

$$\lambda^d \xi \in \Lambda_+ \mathfrak{g}_\tau^{\mathbb{C}} \text{ and } \xi \in \Lambda \mathfrak{g}_\tau.$$

But $b \in \Lambda_+ G_\tau^{\mathbb{C}}$, so $\lambda^d b \xi_o b^{-1} \in \Lambda_+ \mathfrak{g}_\tau^{\mathbb{C}}$. Also $\Phi^\lambda \in \Lambda G_\tau$, so $\Phi^{\lambda^{-1}} \xi_o \Phi^\lambda \in \Lambda \mathfrak{g}_\tau$. Part (ii) follows easily after collecting the following facts (letting $\eta_o := \lambda^{d-1} \xi_o$)

① $\Phi^{-1} d\Phi = \lambda^{-1} \eta_1 dz + (\eta_0 dz)_{\mathfrak{k}} + \lambda \bar{\eta}_1 d\bar{z}$ (this is 2.4).

② $\forall \eta \in \mathfrak{k}^{\mathbb{C}}$ $(\eta dz)_{\mathfrak{k}} = (\eta_{\bar{\mathfrak{n}}} + \frac{1}{2}\eta_{\mathfrak{h}}) dz + \overline{(\eta_{\bar{\mathfrak{n}}} + \frac{1}{2}\eta_{\mathfrak{h}})} d\bar{z}$
(see proof below, we apply this to $\eta = \eta_0$).

③ $r(\eta) := \eta_{\bar{\mathfrak{n}}} + \frac{1}{2}\eta_{\mathfrak{h}}$ (this is 2.1).

④ $\xi = \lambda^{1-d} \eta$ (by definition, thus $\eta_1 = \xi_d$ & $\eta_0 = \xi_{d-1}$).

For part (iii), we simply differentiate $\xi$ and use $\Phi(0) = I$.

($\Rightarrow$) If $\Phi^\lambda$ is finite type, with $\xi$ its Killing field (with $\xi(0) = \xi_o$), then let $a := \exp(z \lambda^{d-1} \xi_o)_u$. Note again that $da = a \alpha_\xi$ with $a(0) = I$. Thus, by the uniqueness theory of ODE's, $\Phi^\lambda = a$ and $\Phi^\lambda$ is finite-Symes type. □

Proof of ②. This is equation (2.5) p.236 [BP]. Note that $(\eta dz)_{\mathfrak{k}} = Re(\eta dz)$ and that $\eta_{\bar{\mathfrak{n}}} = \overline{\eta_{\mathfrak{n}}}$. We calculate.

$$\eta dz = (\eta_{\mathfrak{n}} + \eta_{\mathfrak{h}} + \eta_{\bar{\mathfrak{n}}}) dz = (\eta_{\mathfrak{n}} + \eta_{\mathfrak{h}} + \eta_{\bar{\mathfrak{n}}})(dx + i dy).$$

Thus,

$$\begin{aligned} LHS &= (\eta dz)_{\mathfrak{k}} = Re(\eta dz) \\ &= \frac{1}{2}((\eta_{\mathfrak{n}} + \eta_{\mathfrak{h}} + \eta_{\bar{\mathfrak{n}}}) + (\overline{\eta_{\mathfrak{n}}} + \overline{\eta_{\mathfrak{h}}} + \overline{\eta_{\bar{\mathfrak{n}}}}))dx + \frac{i}{2}((\eta_{\mathfrak{n}} + \eta_{\mathfrak{h}} + \eta_{\bar{\mathfrak{n}}}) - (\overline{\eta_{\mathfrak{n}}} + \overline{\eta_{\mathfrak{h}}} + \overline{\eta_{\bar{\mathfrak{n}}}}))dy \\ &= (\frac{1}{2}(\eta_{\mathfrak{n}} + \overline{\eta_{\mathfrak{n}}}) + \frac{1}{2}(\eta_{\bar{\mathfrak{n}}} + \overline{\eta_{\bar{\mathfrak{n}}}}) + \frac{1}{2}(\eta_{\mathfrak{h}} + \overline{\eta_{\mathfrak{h}}}))dx \\ &\quad + (\frac{i}{2}(\eta_{\mathfrak{n}} - \overline{\eta_{\mathfrak{n}}}) + \frac{i}{2}(\eta_{\bar{\mathfrak{n}}} - \overline{\eta_{\bar{\mathfrak{n}}}}) + \frac{i}{2}(\eta_{\mathfrak{h}} - \overline{\eta_{\mathfrak{h}}}))dy. \end{aligned}$$

Meanwhile,

$$
\begin{aligned}
RHS &= (\eta_{\bar{\mathfrak{n}}} + \frac{1}{2}\eta_{\mathfrak{h}})dz + \overline{(\eta_{\bar{\mathfrak{n}}} + \frac{1}{2}\eta_{\mathfrak{h}})}d\bar{z} \\
&= (\frac{1}{2}(\eta_{\bar{\mathfrak{n}}} + \overline{\eta_{\bar{\mathfrak{n}}}}) + \frac{1}{4}(\eta_{\mathfrak{h}} + \overline{\eta_{\mathfrak{h}}}))(dz + d\bar{z}) \\
&\quad + (\frac{1}{2}(\eta_{\bar{\mathfrak{n}}} - \overline{\eta_{\bar{\mathfrak{n}}}}) + \frac{1}{4}(\eta_{\mathfrak{h}} - \overline{\eta_{\mathfrak{h}}}))(dz - d\bar{z}) \\
&= ((\eta_{\bar{\mathfrak{n}}} + \overline{\eta_{\bar{\mathfrak{n}}}}) + \frac{1}{2}(\eta_{\mathfrak{h}} + \overline{\eta_{\mathfrak{h}}}))dx \\
&\quad + i((\eta_{\bar{\mathfrak{n}}} - \overline{\eta_{\bar{\mathfrak{n}}}}) + \frac{1}{2}(\eta_{\mathfrak{h}} - \overline{\eta_{\mathfrak{h}}}))dy.
\end{aligned}
$$

Since $\eta_{\bar{\mathfrak{n}}} = \overline{\eta_{\mathfrak{n}}}$ we have $LHS = RHS$.

# 3 Lorentz Harmonic Maps into $S^2(1)$

We believe that the results of this section hold with little modification for Lorentz harmonic maps into a symmetric space (or for that matter into a k-symmetric space). However several of the tools we apply have been proved only for the case of $S^2(1)$ and we shall therefore restrict to this case. We also mention that all the results remain true for weakly regular Lorentz harmonic maps. In that setting it is possible to study global questions. Thus all the results should otherwise be considered as local. It is unknown what happens in the non-weakly regular cases.

## 3.1 Loop Spaces

We therefore let $N = S^2(1)$. We consider $G = S^3(1) = SU(2) =$ unit quaternions, and $K = S^1$. The map $\tau$ is given by $Ad\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. (See [MS1] for details.) Alternatively ([T]), $G = SO(3)$, $K = SO(2)$ and $\tau$ is given by $Ad\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

When studying Lorentz harmonic maps $\phi : \mathbb{R}^{1,1} \to S^2(1)$ we need "loop" spaces where the variable $\lambda \in \mathbb{R}_*$. The $H^s$ norm is replaced by a norm introduced in [T] and we consider the completion of the following spaces with respect to that norm.

$$\begin{aligned}
\Lambda\mathfrak{g}_\tau &= \{\xi = \sum_{n\in\mathbb{Z}} \xi_n\lambda^n : \xi_n \in \mathfrak{g} \text{ and } \tau\xi_n = (-1)^n\xi_n \ \forall n\};\\
\Lambda_+\mathfrak{g}_\tau &= \{\xi \in \Lambda\mathfrak{g}_\tau : \xi_n = 0 \text{ for } n < 0\};\\
\Lambda_-\mathfrak{g}_\tau &= \{\xi \in \Lambda\mathfrak{g}_\tau : \xi_n = 0 \text{ for } n > 0\};\\
\Lambda_{+_*}\mathfrak{g}_\tau &= \{\xi \in \Lambda_+\mathfrak{g}_\tau : \xi(0) = 0\};\\
\Lambda_{-_*}\mathfrak{g}_\tau &= \{\xi \in \Lambda_-\mathfrak{g} : \xi(\infty) = 0\}.
\end{aligned}$$

The corresponding "loop" groups are denoted by $\Lambda G_\tau$, $\Lambda_+ G_\tau$, $\Lambda_- G_\tau$, $\Lambda_{+_*} G_\tau$ and $\Lambda_{-_*} G_\tau$.

At the loop algebra level the (twisted) Birkoff vector space decomposition is given by :

$$\Lambda\mathfrak{g}_\tau = \Lambda_+\mathfrak{g}_\tau \oplus \Lambda_{-_*}\mathfrak{g}_\tau \text{ (resp. } \Lambda_\tau = \Lambda_-\mathfrak{g}_\tau \oplus \Lambda_{+_*}\mathfrak{g}_\tau);$$

and at the loop group level the (twisted) Birkoff decomposition is given (via multiplication) by a diffeomorphism onto an open dense set of products [T]:

$$\Lambda_+ G_\tau \times \Lambda_{-_*} G_\tau \to \Lambda_+ G_\tau \cdot \Lambda_{-_*} G_\tau \subset \Lambda G_\tau$$

$$\text{(resp. } \Lambda_- G_\tau \times \Lambda_{+_*} G_\tau \to \Lambda_- G_\tau \cdot \Lambda_{+_*} G_\tau \subset \Lambda G_\tau).$$

We will denote the decomposition of an element $g \in \Lambda G_\tau^{\mathbb{C}}$ via

$$g = g_{+_\ell} g_{-_r} \text{(resp. } g = g_{-_\ell} g_{+_r}).$$

Finally, again let

$$\Lambda_d := \{\xi \in \Lambda\mathfrak{g}_\tau : \xi_n = 0 \text{ for all } |n| > d\}.$$

## 3.2 Maps

$M$, which is a Riemann surface in the harmonic case, is now a simply connected domain in $\mathbb{R}^{1,1}$ and we consider maps $\phi : M \subset \mathbb{R}^{1,1} \to N = S^2(1)$. Again a framing of $\phi$ is a map $\Phi : M \subset \mathbb{R}^{1,1} \to G = S^3(1)$ such that $\pi\circ\Phi = \phi$. Such lifts always exist globally. Let $\Phi$ be a framing of $\phi$, $\alpha := \Phi^{-1}d\Phi$ and $(\lambda \in \mathbb{R} - \{0\})$

$$\alpha^\lambda := \lambda\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda^{-1}\alpha''_{\mathfrak{m}} \quad (= \lambda\tilde{\alpha}'_{\mathfrak{m}}dx + \alpha_{\mathfrak{k}} + \lambda^{-1}\tilde{\alpha}''_{\mathfrak{m}}dy), \tag{3.1}$$

where $\alpha_{\mathfrak{m}}$ decomposes into its $dx$ and $dy$-parts $\alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$. A standard result is that if $\phi$ is Lorentz harmonic then $\alpha^\lambda$ satisfies the Zero Curvature Equation (ZCE)

$$d\alpha^\lambda + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0.$$

Conversely, if $\alpha^\lambda$ is a loop of 1-forms of the form (3.1), then $d\Phi^\lambda = \Phi^\lambda \alpha^\lambda$ can be (smoothly) integrated and projected to yield a loop of Lorentz harmonic maps to $N$.

A map $a : M \to \Lambda G_\tau$ is called an extended framing if

$$a^{-1}da = \lambda\alpha'_{\mathfrak{m}} + \alpha_0 + \lambda^{-1}\alpha''_{\mathfrak{m}}.$$

Note that any extended framing gives rise to a loop of Lorentz harmonic maps (since $\alpha^\lambda := a^{-1}da$ satisfies the ZCE). Also note again that any Lorentz harmonic map on $M$ defines an extended framing (since $da = a\alpha^\lambda$ can be smoothly integrated).

Let $d \equiv 1 \bmod 2$. If $\xi : M \to \Lambda_d$ then $\alpha_\xi^+$ (resp. $\alpha_\xi^-$), a $\Lambda\mathfrak{g}_\tau$-valued 1-from on $M$, is defined by

$$\alpha_\xi^+ = \lambda\xi_d dx + \xi_{d-1}dx + \lambda^{-1}\xi_{-d}dy,$$

(resp. $\alpha_\xi^- = \lambda\xi_d dx - \xi_{1-d}dy + \lambda^{-1}\xi_{-d}dy$). It is pivotal to the theory that such an $\alpha_\xi^+$ (resp. $\alpha_\xi^-$) automatically satisfies the ZCE as soon as $\xi$ satisfies $d\xi = [\xi, \alpha_\xi^+]$ (resp. $d\xi = [\xi, \alpha_\xi^-]$). See for example p.82-83 of [MS1].

Alternatively one could use the normalization in [MS1]:

$$\alpha_\xi = (\lambda\xi_d + \frac{1}{2}\xi_{d-1})dx - (\lambda^{-1}\xi_{-d} + \frac{1}{2}\xi_{1-d})dy.$$

However this choice is not (as naturally) compatible with the normalization suggested by [DPW] and [T].

## 3.3 Main Result

A map $\Phi^\lambda : M \longrightarrow \Lambda G_\tau$ is called finite type if $d\Phi^\lambda = \Phi^\lambda \alpha_\xi^+$ , $\Phi^\lambda(0) = I$ where $\xi : M \to \Lambda_d$ and $d\xi = [\xi, \alpha_\xi^+]$ , $\xi(0) = \xi_o$ for some $\xi_o \in \Lambda_d$ where $d \equiv 1 \bmod 2$. The loop $\xi$ is called the Killing field of $\Phi^\lambda$. It is a well-known result that every $\xi_o \in \Lambda_d$ with $d \equiv 1 \bmod 2$ yields such a map of finite type (again see, for example, [MS1]). The function $\det(\xi(\lambda) - \mu I)$ is called the spectral curve of $\Phi^\lambda$ and does not depend on $(x,y)$. A harmonic map $\phi : M \longrightarrow S^2(1)$ is called finite type if $\phi = \pi \circ ev_1 \circ \Phi^\lambda$ where $\Phi^\lambda : M \longrightarrow \Lambda G_\tau$ is finite type. (Again $ev_1$ is the evaluation map.)

A map $a : M \longrightarrow \Lambda G_\tau$ is called finite-Symes type if $a = g^y g_{+_\ell} = g^x g_{-_r}^{-1}$ where $g^y := \exp(y\lambda^{d-1}\xi_o)$, $g^x := \exp(x\lambda^{-d+1}\xi_o)$, and $g := \exp(x\lambda^{-d+1}\xi_o - y\lambda^{d-1}\xi_o) = g^{y^{-1}} g^x$ for some $\xi_o \in \Lambda_d$ where $d \equiv 1 \bmod 2$. Note $g(0,0) = I$ and recall $g = g_{+_\ell} g_{-_r}$. Thus $g_{+_\ell}(0,0) = g_{-_r}(0,0) = I$ by the uniqueness of the Birkhoff factorization. Note also that $g^y(0,0) = g^x(0,0) = I$ and hence $a(0,0) = I$. We note that every $\xi_o \in \Lambda_d$ with $d \equiv 1 \bmod 2$ yields a finite-Symes type map: we define $g^y$, $g^x$ and $g$ from $\xi_o$ as above and we consider the Birkhoff splitting of $g$. Then $a = g^y g_{+_\ell} = g^x g_{-_r}^{-1}$ is a map of finite-Symes type. A Lorentz harmonic map $\phi : \mathbb{R}^2 \longrightarrow G/K$ is called finite-Symes type if $\phi = \pi \circ ev_1 \circ a$ where $a : M \longrightarrow \Lambda G_\tau$ is finite-Symes type.

**Lemma 3.1.** *If $a$ is finite-Symes type, then $a$ is an extended framing and $a^{-1}da$ is of the form*

$$a^{-1}da = \lambda A_1 dx + (A_0 dx + B_0 dy) + B_{-1}\lambda^{-1} dy. \tag{3.2}$$

**Proof:** In this proof, let $g_+ := g_{+_\ell}$ and $g_- := g_{-_r}^{-1}$. We follow [T].

$$\begin{aligned} a^{-1}da &= (g_+^{-1} g^{y^{-1}})(g^y dg_+ + dg^y g_+), && (3.3)\\ &= g_+^{-1} dg_+ + g_+^{-1}\lambda^{d-1}\xi_0 dy\, g_+ \,. && (3.4)\end{aligned}$$

$\lambda a^{-1}da$ contains only non-negative terms and the constant term has no dx-part. Similarly (using $a = g^x g_-$) we see $\lambda^{-1}a^{-1}da$ contains only non-positive terms and the constant term has no dy-part and we have a formula similar to 3.4 but with $g_-$:

$$\begin{aligned} a^{-1}da &= (g_-^{-1} g^{x^{-1}})(g^x dg_- + dg^x g_-), && (3.5)\\ &= g_-^{-1} dg_- + g_-^{-1}\lambda^{-d+1}\xi_0 dx\, g_- \,. && (3.6)\end{aligned}$$

The lemma follows directly from this. □

**Theorem 3.2.** *A Lorentz harmonic map $\phi : M \longrightarrow G/K$ is finite type if and only if it is finite-Symes type.*

**Proof:** ($\Leftarrow$) Let $\xi = \Phi^{\lambda^{-1}}\xi_o\Phi^\lambda$. We need to show that

i) $\xi : M \longrightarrow \Lambda_d$,

ii) $\Phi^{\lambda^{-1}} d\Phi^{\lambda} = \lambda \xi_d dx + \xi_{d-1} dx + \lambda^{-1} \xi_{-d} dy$,

iii) $d\xi = [\xi, \Phi^{\lambda^{-1}} d\Phi^{\lambda}]$ with $\xi(0) = \xi_o$.

For part (i), for shorthand, we again let $g_+ := g_{+\ell}$ and $g_- := g_{-r}^{-1}$. Thus

$$
\begin{aligned}
\Phi^{\lambda^{-1}} \xi_o \Phi^{\lambda} &= g_-^{-1} g_x^{-1} \xi_o g_x g_- \\
&= g_-^{-1} \xi_o g_-
\end{aligned}
$$

Similarly $\Phi^{\lambda^{-1}} \xi_o \Phi^{\lambda} = g_+^{-1} \xi_o g_+$. Since $g_+, g_+^{-1}$ (resp. $g_-, g_-^{-1}$) have only nonnegative (resp. nonpositive) terms, we conclude $\xi \in \Lambda_d$ .

Part (ii). Since $\Phi^{\lambda}$ is finite-Symes type we can apply equations 3.4 and 3.6 with $a$ replaced with $\Phi^{\lambda}$. Thus we have:

$$
\Phi^{\lambda^{-1}} d\Phi = g_+^{-1} dg_+ + g_+^{-1} \lambda^{d-1} \xi_o dy \, g_+ \tag{3.7}
$$

$$
\Phi^{\lambda^{-1}} d\Phi = g_-^{-1} dg_- + g_-^{-1} \lambda^{1-d} \xi_o dx \, g_- . \tag{3.8}
$$

Next we note that this yields

$$
\Phi^{\lambda^{-1}} d\Phi^{\lambda} = Ad\Phi^{\lambda^{-1}} \lambda^{-d+1} \xi_o dx + g_-^{-1} g_{-_x} dx + g_-^{-1} g_{-_y} dy, \tag{3.9}
$$

$$
\Phi^{\lambda^{-1}} d\Phi^{\lambda} = Ad\Phi^{\lambda^{-1}} \lambda^{d-1} \xi_o dy + g_+^{-1} g_{+_x} dx + g_+^{-1} g_{+_y} dy. \tag{3.10}
$$

To derive 3.9 and 3.10 we use 3.7 and 3.8 and the following easy facts:

① $dg_- = g_{-_x} dx + g_{-_y} dy$,

② $dg_+ = g_{+_x} dx + g_{+_y} dy$,

③ $g^{x^{-1}} \lambda^{-d+1} \xi_o g^x = \lambda^{-d+1} \xi_o$,

④ $g^{y^{-1}} \lambda^{d-1} \xi_o g^y = \lambda^{d-1} \xi_o$.

By Lemma 3.1 we need only consider the terms $(\Phi^{\lambda^{-1}} d\Phi^{\lambda})_1$, $(\Phi^{\lambda^{-1}} d\Phi^{\lambda})_0$, and $(\Phi^{\lambda^{-1}} d\Phi^{\lambda})_{-1}$. This yields six equations:

$$
(\Phi^{\lambda^{-1}} d\Phi^{\lambda})_1 = \xi_d dx + 0 dy, \tag{3.11}
$$

$$
(\Phi^{\lambda^{-1}} d\Phi^{\lambda})_1 = \xi_{2-d} dy + (g_+^{-1} g_{+_x})_1 dx + (g_+^{-1} g_{+_y})_1 dy, \tag{3.12}
$$

$$
(\Phi^{\lambda^{-1}} d\Phi^{\lambda})_0 = \xi_{d-1} dx + (g_-^{-1} g_{-_x})_0 dx + (g_-^{-1} g_{-_y})_0 dy, \tag{3.13}
$$

$$
(\Phi^{\lambda^{-1}} d\Phi^{\lambda})_0 = \xi_{1-d} dy + (g_+^{-1} g_{+_x})_0 dx + (g_+^{-1} g_{+_y})_0 dy, \tag{3.14}
$$

$$
(\Phi^{\lambda^{-1}} d\Phi^{\lambda})_{-1} = \xi_{d-2} dx + (g_-^{-1} g_{-_x})_{-1} dx + (g_-^{-1} g_{-_y})_{-1} dy, \tag{3.15}
$$

$$
(\Phi^{\lambda^{-1}} d\Phi^{\lambda})_{-1} = 0 dx + \xi_{-d} dy \tag{3.16}
$$

From 3.11 we have $(\Phi^{\lambda^{-1}} d\Phi^{\lambda})_1 = \xi_d dx$. From 3.16 we have $(\Phi^{\lambda^{-1}} d\Phi^{\lambda})_{-1} = \xi_{-d} dy$. Recalling that $g_{-0_x} = 0$ and $g_{-0_y} = 0$, from 3.13 we have $(\Phi^{\lambda^{-1}} d\Phi^{\lambda})_0 = \xi_{d-1} dx$.

The proof of iii) is the same as the corresponding part in the harmonic case.

($\Rightarrow$) If $\Phi^{\lambda}$ is finite type with $\xi$ its Killing field and $\xi(0) = \xi_o$, then $\xi_o$ is in $\Lambda_d$ and we define $a := g^y g_{+\ell}$ as in front of Lemma 3.1. Then the first part of the proof is applicable, replacing $\Phi^{\lambda}$ by $a$ and we see that both, $\tilde{\xi} = a^{-1}\xi_o a$ and $\xi = \Phi^{\lambda^{-1}} \xi_o \Phi^{\lambda}$ satisfy (iii). Hence $\xi = \tilde{\xi}$ and $\alpha_{\xi} = \alpha_{\tilde{\xi}}$ follows. Therefore, (ii) has the unique solution $\Phi^{\lambda} = a$, since both functions satisfy the same differential equation and have the same initial condition. Therefore $\Phi^{\lambda}$ is of finite-Symes type.$\square$

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