

MEDIAN VALUES, 1-HARMONIC FUNCTIONS, AND FUNCTIONS OF LEAST GRADIENT

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ABSTRACT. Motivated by the mean value property of harmonic functions, we introduce the local and global median value properties for continuous functions of two variables. We show that the Dirichlet problem associated with the local median value property is either easy or impossible to solve, and we prove that continuous functions with this property are 1-harmonic in the viscosity sense. We then close with the following conjecture: a continuous function having the global median value property and prescribed boundary values coincides with the function of least gradient having those same boundary values.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a domain with smooth boundary $\partial\Omega$, let $B(x, r)$ denote a ball with radius r , center x , and boundary $\partial B(x, r)$, and let $\int_E f d\mu$ denote the average of f over the set E with respect to the measure μ . It is well-known that the continuous function u is harmonic in Ω if and only if

$$u(x) = \int_{\partial B(x,r)} u(s) ds \quad \text{whenever } x \in \Omega \quad \text{and} \quad B(x, r) \Subset \Omega; \quad (1)$$

this is the famous mean value property of harmonic functions [4]. As is also well-known, imposing a Dirichlet boundary condition determines a unique solution of the functional equation (1). This equation has been well understood for a long time, and it is natural to wonder what happens if we replace the averaging operator in (1) with a different statistical measure.

What happens, for example, if we replace the average in (1) with the median of u over spheres centered at x ? Requiring u to be continuous makes this question meaningful; as discussed below, medians are not well-defined for functions that are merely measurable or integrable.

When $N = 2$, which we assume henceforth, we can use elementary tools to analyze continuous solutions of the functional equation

$$u(x) = \operatorname{median}_{s \in \partial B(x,r)} \{ u(s) \} \quad \text{for } x \in \Omega \quad \text{and} \quad 0 < r \leq R(x), \quad (2)$$

where $R : \Omega \rightarrow \mathbb{R}$ is a continuous function such that

$$0 < R(x) \leq \text{dist}(x, \partial\Omega).$$

Since $R(x)$ can be strictly smaller than the distance $\text{dist}(x, \partial\Omega)$ from x to $\partial\Omega$, equation (2) defines the *local median value property*, a property that does not seem to have been studied before. We will show that equation (2) is closely related to a nonlinear partial differential equation (PDE) that has a reputation for being difficult. The degeneracy of this particular PDE precludes direct applications of standard techniques, making the basic problems of existence, uniqueness, and regularity of its solutions rather thorny. Our analysis of equation (2) clarifies some of these issues: if we impose a Dirichlet boundary condition on solutions of (2), we will see that we should generally expect either nonexistence or nonuniqueness, depending on the interaction of the boundary data and the geometry of $\partial\Omega$.

The mean value property (1) of harmonic functions is global, in the sense that it holds at $x \in \Omega$ for any ball $B(x, r)$ that is strictly contained in Ω . The global version of (2) is also of interest; functions with the *global median value property* satisfy the functional equation

$$u(x) = \text{median}_{s \in \partial B(x,r)} \{ u(s) \} \quad \text{whenever } x \in \Omega \text{ and } B(x, r) \Subset \Omega. \quad (3)$$

Combining equation (3) with a Dirichlet boundary condition yields a problem that is harder to solve than its local counterpart. We conjecture that the solutions of this global problem are precisely the functions of least gradient, for which existence and uniqueness results are only available when Ω is strictly convex [15].

2. THE LOCAL MEDIAN VALUE PROPERTY

If f is a measurable function with respect to the measure μ , then m is a median of f over the measurable set E if and only if

$$\mu\{x \in E : f(x) \geq m\} \geq \frac{\mu(E)}{2} \quad \text{and} \quad \mu\{x \in E : f(x) \leq m\} \geq \frac{\mu(E)}{2}.$$

It is easy to see that such a median m need not be unique. If f is integrable, then m is a median of f over E if and only if [13]

$$\int_E |f(x) - m| d\mu(x) = \min_{y \in \mathbb{R}} \left\{ \int_E |f(x) - y| d\mu(x) \right\},$$

but integrability still does not guarantee a unique median. Using this variational characterization, however, Noah proved in [10] that continuity suffices to determine median values uniquely. By different methods, Waksman and Wasilewsky proved the same result in [14] for continuous functions on planar domains. By working exclusively with continuous functions, we are

therefore assured that medians over spheres are well-defined and that the functional equations (2) and (3) make sense.

To understand (2), we first observe that any continuous function whose level sets are monotonically ordered straight lines has the local median value property. On the other hand, if u is a smooth function such that $Du(x_0) \neq 0$ at $x_0 \in \Omega$, then the Implicit Function Theorem guarantees that, on some sufficiently small ball $B(x_0, r)$, the level sets of u are smooth curves whose levels vary monotonically; if u also satisfies (2), the level curve through x_0 must be a diameter of $B(x_0, r)$. The level sets of a smooth solution of (2) must therefore be straight lines wherever its gradient does not vanish. Our first result, Theorem 1, eliminates this smoothness assumption. Before going through its proof, we note the following obvious and useful result, a weak maximum principle for solutions of (2).

Proposition 1. *A nonconstant continuous solution of (2) cannot have strict interior maxima or minima.*

Figure 1 illustrates a variant of this maximum principle that will be essential below. It depicts a continuous function u that has a “ridge line,” a curve along which $u = a$ such that $u < a$ on either side (of course, u could just as well be larger than a on either side of this curve). Our proof of Theorem 1 relies on the fact that such a situation cannot hold if u has the local median value property everywhere.

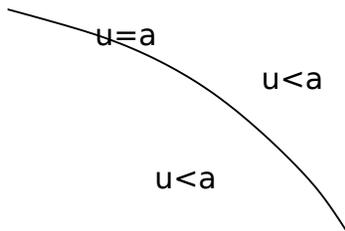


FIGURE 1. A continuous function cannot satisfy (2) along a ridge line.

Theorem 1. *Let u be a nonconstant continuous solution of (2). If K is a path-connected component of some level set of u , then $\partial K \cap \Omega$ consists of line segments with endpoints on $\partial\Omega$.*

Proof. Let K be a path-connected component of the level set $u^{-1}(a)$, and let $x \in K$. The local median value property guarantees that points in K cannot be isolated; consequently, at least one nontrivial path must pass through x .

Step 1. Suppose that, for some $r > 0$, $\overline{B(x, r)} \cap u^{-1}(a)$ consists of one simple path Γ with endpoints $y, z \in \partial B(x, r)$. Decrease r (if necessary) so that (2) holds on $\partial B(x, r)$. By the continuity of u and the maximum principle, y and z must divide $\partial B(x, r)$ into two open arcs such that $u > a$ on one of them and $u < a$ on the other. The local median value property forces these two arcs to have the same length; y and z are therefore antipodal points and Γ must be a diameter of $B(x, r)$.

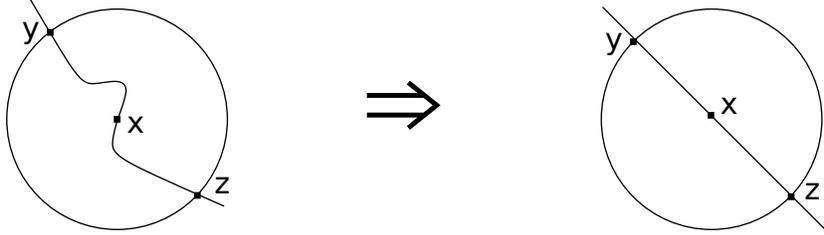


FIGURE 2. A simple path through x must be straight.

Step 2. Now suppose that, for some $r > 0$, $\overline{B(x, r)} \cap u^{-1}(a) \setminus \{x\}$ consists of simple paths that are not intersected by any other paths. It follows from the preceding argument that these paths must be line segments emanating from x (Figure 3).

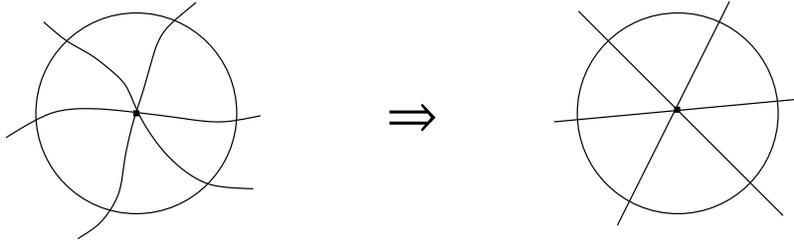


FIGURE 3. Separable paths through x must be straight.

Continuity and the maximum principle prohibit an odd number of line segments from meeting at x , since u must be alternately larger than a and smaller than a in the sectors determined by these segments. Moreover, four or more segments cannot meet at x : if the angle between two adjacent segments were less than or equal to $\pi/2$, then the local median value property would be violated (see Figure 4). Applying Step 1, we conclude that the only path through x inside $B(x, r)$ is a diameter of $B(x, r)$.

Step 3. If the paths emanating from x can be separated, then the previous step shows that, on some sufficiently small neighborhood of x , there can be only one straight path through x . In fact, the observations in Step 2 show

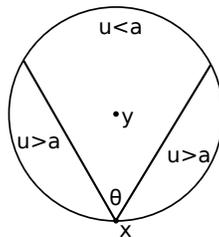


FIGURE 4. If θ were less than $\pi/2$, then (2) would force $u(y) > a$, a contradiction; if θ were exactly $\pi/2$, then we would have $u(y) = a$, another contradiction. Consequently, $\theta > \pi/2$.

that, if two of the paths emanating from x cannot be separated, then K must have a nonempty interior. Consequently, if the interior of K is empty, K must be a line segment. By the local median value property, the endpoints of this segment cannot be in Ω ; its endpoints must be on $\partial\Omega$. The theorem therefore holds for path-connected components with no interior.

Step 4. Finally, suppose that the interior of K is nonempty, pick a point $y \in K$, and let z be a point in K such that the segment L connecting y and z is inside Ω . If L did not belong to K , then the continuity of u would guarantee the existence of a point p on L whose level set component had empty interior. By the previous step, that component would be a line segment with endpoints on the boundary of Ω , separating y from z and contradicting the fact that y and z belong to K . This completes the proof. □

3. THE DIRICHLET PROBLEM

We now seek a continuous solution u of (2) that has prescribed values along the boundary of Ω . Theorem 1 tells us how to go about solving the resulting Dirichlet problem: roughly speaking, the level sets of u need to be line segments or polygons.

Figure 5 illustrates a solution of this problem for a nonconvex domain Ω and very particular boundary data. This example is artificial, however; in general, we cannot expect to solve the Dirichlet problem on a region Ω that is not strictly convex, precisely because we cannot connect arbitrary pairs of boundary points with line segments that remain inside Ω . See, for instance, Figure 6.

If Ω is strictly convex, on the other hand, solving the Dirichlet problem is much easier. As an example, let $\Omega \subset \mathbb{R}^2$ be the open unit ball, and let $g(x, y) = |y|$ for $(x, y) \in \partial\Omega$. It is easy to see that, for any $\alpha \in (0, 1)$, the

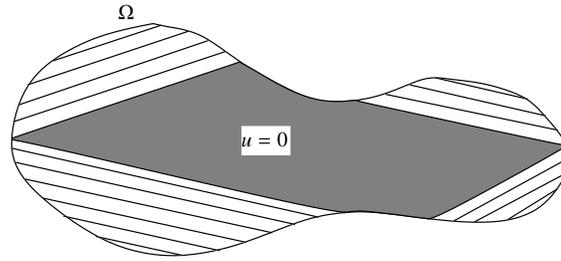


FIGURE 5. A solution of the Dirichlet problem on a nonconvex domain Ω .

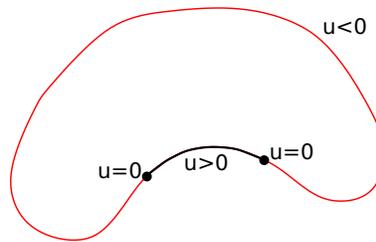


FIGURE 6. The Dirichlet problem cannot necessarily be solved if Ω is not strictly convex. Here, we cannot connect positive boundary values with line segments inside Ω .

function

$$u_\alpha(x, y) := \begin{cases} |y| & \text{when } |y| \geq \alpha, \\ \sqrt{1-x^2} & \text{when } |y| < \alpha \text{ and } |x| \geq \sqrt{1-\alpha^2}, \\ \alpha & \text{otherwise} \end{cases} \quad (4)$$

has the local median value property and satisfies $u = g$ on $\partial\Omega$. Figure 7 provides contour plots of solutions for three values of α , illustrating the fact that different solutions of the Dirichlet problem correspond to the different ways of connecting the boundary values with straight lines.

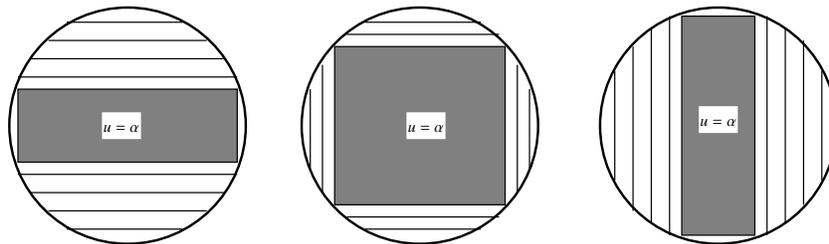


FIGURE 7. Solutions u_α of the Dirichlet problem on the unit ball with $u_\alpha(x, y) = |y|$ on its boundary. From left to right, $\alpha < \sqrt{2}/2$, $\alpha = \sqrt{2}/2$, and $\alpha > \sqrt{2}/2$.

An existence result for reasonable boundary data follows directly. Specifically, let Ω be strictly convex and bounded, and suppose that $g : \partial\Omega \rightarrow \mathbb{R}$ is a nonconstant continuous function whose level sets have finitely many endpoints. (If g is constant, it follows from the maximum principle that the only solution of the Dirichlet problem is constant.) By parametrizing $\partial\Omega$, we can regard g as a continuous 2π -periodic function, $g : [0, 2\pi] \rightarrow \mathbb{R}$. Define m and M to be the minimum and maximum of g , respectively, pick any $\lambda \in (m, M)$, and define $\Lambda := g^{-1}(\lambda)$, the set of angles θ such that $g(\theta) = \lambda$. Since Λ has finitely many endpoints, $\partial\Lambda = \{\theta_1, \dots, \theta_k\}$ for some k , with $\theta_1 < \theta_2 < \dots < \theta_k$.

Construct a generalized polygon whose vertices on $\partial\Omega$ correspond to the angles $\theta_1, \dots, \theta_k$. This is a generalized polygon since some of its edges could be curved portions of $\partial\Omega$; this will happen whenever $g(\theta) \equiv \lambda$ on the closed interval $[\theta_j, \theta_{j+1}]$. As for the other edges, we can proceed as follows. Suppose, without loss of generality, that $g(\theta) > \lambda$ for $\theta_j < \theta < \theta_{j+1}$. For sufficiently small $\varepsilon > 0$, the level set $g^{-1}(\lambda + \varepsilon) \cap [\theta_j, \theta_{j+1}]$ will consist of exactly two angles, which we connect with a line segment. If this level set never contains more than two angles as ε increases, then we are done with this sector of Ω . Otherwise, there will be a smallest $\varepsilon > 0$ such that the boundary of $g^{-1}(\lambda + \varepsilon) \cap [\theta_j, \theta_{j+1}]$ consists of angles ϕ_1, \dots, ϕ_ℓ for $\ell \geq 3$. Construct a generalized polygon with vertices corresponding to these ℓ angles, and then repeat this procedure in each resulting sector of Ω . The compactness of $\overline{\Omega}$ guarantees that this process will terminate, yielding a solution of the Dirichlet problem; its continuity follows directly from the continuity of g . We have thus proven

Theorem 2. *If $\Omega \subset \mathbb{R}^2$ is a strictly convex, bounded open set and $g : \partial\Omega \rightarrow \mathbb{R}$ is a nonconstant continuous function whose level sets have finitely many endpoints, then there is at least one continuous function u that satisfies (2) and equals g on $\partial\Omega$.*

Implementing this procedure for different values of λ can produce different solutions; in Figure 7 above, the three solutions correspond to choosing $\lambda = \alpha$ for three different values of α . Consequently, we generally expect to find multiple solutions when Ω is strictly convex, and an intriguing problem for future work is to quantify the number of solutions that correspond to a given boundary condition. For a domain Ω that is not strictly convex, it would also be interesting to characterize those boundary conditions for which solutions exist.

4. 1-HARMONIC FUNCTIONS

For $p > 1$, the continuous function u is p -harmonic if and only if it is a viscosity solution of

$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = 0. \quad (5)$$

Juutinen et al. proved [8] that this is equivalent to the more common definition of p -harmonic functions as weak solutions of (5), the Euler-Lagrange equation corresponding to the variational integral

$$v \mapsto \int_{\Omega} |Dv|^p dx \quad \text{for } v \in W^{1,p}(\Omega). \quad (6)$$

When $p > 1$, the variational approach to (5) is especially elegant, since the associated functional (6) is strictly convex and weakly lower semicontinuous and the Sobolev space $W^{1,p}(\Omega)$ is reflexive.

Formally setting $p = 1$, one can define a 1-harmonic function u to be a viscosity solution of

$$-\operatorname{div}\left(\frac{Du}{|Du|}\right) = 0,$$

an equation that is not nearly as well understood as (5). It is more complicated than (5) in several regards; the functional (6) is no longer strictly convex, for example, and $W^{1,1}(\Omega)$ is not as hospitable a setting for functional analysis as $W^{1,p}(\Omega)$ (as explained further in, e.g., [3] and [5]).

For various reasons, it is more convenient for us to define u to be 1-harmonic if and only if it is a viscosity solution of

$$-\Delta_1 u := -|Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0. \quad (7)$$

As clarified below in Definition 1, we can loosely interpret this equation to mean that either $|Du| = 0$ or $\operatorname{div}(Du/|Du|) = 0$, and we will see that continuous solutions of (2) are 1-harmonic. We first define viscosity solutions of (7), referring to [8] for more details when $p > 1$.

Definition 1. The continuous function $\bar{u} : \Omega \rightarrow \mathbb{R}$ is a viscosity supersolution of (7) if and only if

- (1) $\bar{u} \not\equiv \infty$ and
- (2) whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ satisfies

$$\begin{cases} \bar{u}(x_0) = \varphi(x_0), \\ \bar{u}(x) > \varphi(x) \quad \text{for } x \neq x_0, \quad \text{and} \\ D\varphi(x_0) \neq 0, \end{cases} \quad (8)$$

φ also satisfies

$$-\Delta_1 \varphi(x_0) \geq 0.$$

The continuous function \underline{u} is a viscosity subsolution of (7) if and only if $-\underline{u}$ is a viscosity supersolution of (7), and u is 1-harmonic in the viscosity sense if u is both a viscosity subsolution and a viscosity supersolution of (7).

The work in [8] motivates this definition, as Juutinen et al. proved that, when $p > 1$, we only need to work with test functions whose gradients do not vanish at the point x_0 of interest. It then follows that our definition of 1-harmonic and the usual one are equivalent.

Theorem 3. *A continuous function $u : \Omega \rightarrow \mathbb{R}$ that satisfies (2) is 1-harmonic in the viscosity sense.*

Proof. Let $x_0 \in \Omega$, and let $\varphi \in C^2(\Omega)$ be a test function that touches u from below at x_0 . Using (2), (8) and the monotonicity of the median, we have

$$\operatorname{median}_{\partial s \in B(x_0, r)} \{ \varphi(s) \} < \operatorname{median}_{s \in \partial B(x_0, r)} \{ u(s) \} = u(x_0) = \varphi(x_0),$$

so that

$$\varphi(x_0) - \operatorname{median}_{s \in \partial B(x_0, r)} \{ \varphi(s) \} > 0. \tag{9}$$

Identity (11) in [6] establishes that

$$\varphi(x_0) - \operatorname{median}_{s \in \partial B(x_0, r)} \{ \varphi(s) \} = -\frac{r^2}{2} \Delta_1 \varphi(x_0) + o(r^2),$$

so dividing by $r^2/2$ and sending $r \rightarrow 0$ in (9) shows that u is a viscosity supersolution of (7). Similarly, u is a viscosity subsolution of (7), proving that u is 1-harmonic in the viscosity sense. \square

The key result above, identity (11) of [6], is the reason for including the prefactor $|Du|$ in our definition of the 1-Laplacian Δ_1 . Not coincidentally, our definition of Δ_1 coincides with the elliptic part of the operator in the widely studied mean curvature equation,

$$u_t - |Du| \operatorname{div} \left(\frac{Du}{|Du|} \right) = 0.$$

Previous work relating this parabolic equation, medians, and median-like operators ([1],[2],[9],[11],[12]) motivated the present work.

5. FUNCTIONS OF LEAST GRADIENT

In a series of papers summarized in [15], Ziemer et al. analyzed functions of least gradient, i.e., functions $v \in BV(\Omega) \cap C(\overline{\Omega})$ with minimal total variation and prescribed values along $\partial\Omega$. They proved, in particular, that a unique function of least gradient exists if Ω is strictly convex and the boundary function $g : \partial\Omega \rightarrow \mathbb{R}$ is continuous. One can assemble this function of

least gradient by constructing its level sets: they are minimal surfaces whose boundaries are determined by the levels of g .

This, of course, is exactly how we constructed solutions of the Dirichlet problem earlier, since one-dimensional minimal surfaces in the plane are lines. Our method did not produce a unique solution, however, as we only considered local conditions; the global constraint of minimizing total variation renders functions of least gradient unique. It seems reasonable, then, that there should be a strong connection between the global median value property (3) and functions of least gradient. To that end, we propose

Conjecture 1. *Suppose that Ω is strictly convex and that $g : \partial\Omega \rightarrow \mathbb{R}$ is continuous, and let u^* be the function of least gradient on Ω that equals g on $\partial\Omega$.*

- (1) *There exists a unique continuous solution u of (3) such that $u = g$ on $\partial\Omega$.*
- (2) *$u = u^*$.*

It is easy to see that this conjecture holds in certain cases. For example, among the family $\{u_\alpha\}$ of solutions of (2) defined by equation (4) earlier, only $u_{1/\sqrt{2}}$ has the global median value property. It follows easily from the coarea formula that this function's total variation is smaller than that of any other u_α , and it is indeed the function of least gradient on the unit circle with this boundary data.

Progress on this conjecture (and on other questions related to the results of this paper) will likely benefit from Juutinen's work [7] on p -harmonic functions and functions of least gradient.

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