PDEs Fall 2014 HW 2

pg 38 #2 Solve $u_{tt} = c^2 u_{xx},\ u(x,0) = \log(1+x^2),\ u_t(x,0) = 4 + x$.

Solution We know that the wave equation has a solution in the form

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

The initial data says

$$\begin{aligned} \phi(x) &= u(x,0) &&= \log(1+x^2) \\ \psi(x) &= u_t(x,0) &&= 4 + x. \end{aligned}$$

Thus,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\log(1+(x+ct)^2) + \log(1+(x-ct)^2)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} 4 + s\, ds \\ &= \frac{1}{2}\left[\log(1+(x+ct)^2) + \log(1+(x-ct)^2)\right] + \frac{1}{2c}\left[8ct + \frac{1}{2}(x+ct)^2 - \frac{1}{2}(x-ct)^2\right] \end{aligned}$$

pg 38 #5 The hammer blow problem. **Solution**Notice that in this problem,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds,$$

where

$$\psi(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } |x| \geq a \end{cases}$$

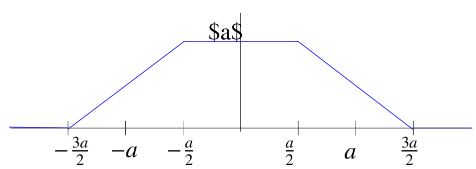
First, we look at all 6 cases for the location of $x+ct$ and $x-ct$ with respect to $-a$ and a. These are organized in the following table:

Case	Interval	u	Picture
1	$x-ct < x+ct < -a < a$	$u(x,t) = \int_{x-ct}^{x+ct} 0\, ds = 0$	$x-ct$ $x+ct$ $-a$ a
2	$-a < a < x-ct < x+ct$	$u(x,t) = \int_{x-ct}^{x+ct} 0\, ds = 0$	$-a$ a $x-ct$ $x+ct$
3	$-a < x-ct < x+ct < a$	$u(x,t) = \int_{x-ct}^{x+ct} 1\, ds = 2ct$	$-a$ $x-ct$ $x+ct$ a
4	$-a < x-ct < a < x+ct$	$u(x,t) = \int_{x-ct}^{a} 1\, ds = a - x + ct$	$-a$ $x-ct$ a $x+ct$
5	$x-ct < -a < x+ct < a$	$u(x,t) = \int_{-a}^{x+ct} 1\, ds = x + ct + a$	$x-ct$ $-a$ $x+ct$ a
6	$x-ct < -a < a < x+ct$	$u(x,t) = \int_{-a}^{a} 1\, ds = 2a$	$x-ct$ $-a$ a $x+ct$

So now we just consider this for each of the given t values:
$t = \frac{a}{2c}$

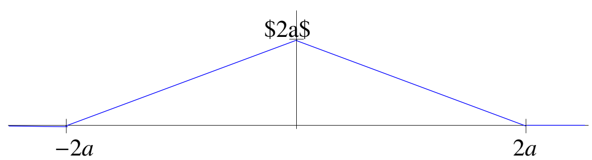
Case	Interval	u
1	$x < -\frac{3a}{2}$	$u(x,t) = \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} 0\ ds = 0$
2	$\frac{3a}{2} < x$	$u(x,t) = \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} 0\ ds = 0$
3	$-\frac{a}{2} < x < \frac{a}{2}$	$u(x,t) = \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} 1\ ds = a$
4	$\frac{a}{2} < x < \frac{3a}{2}$	$u(x,t) = \int_{x-\frac{a}{2}}^{a} 1\ ds = \frac{3a}{2} - x$
5	$-\frac{3a}{2} < x < -\frac{a}{2}$	$u(x,t) = \int_{-a}^{x+\frac{a}{2}} 1\ ds = x + \frac{3a}{2}$
6	$x < -\frac{a}{2}, x > \frac{a}{2}$ $\rightarrow\leftarrow$	

Now we plot u on these intervals to get



$t = \frac{a}{c}$

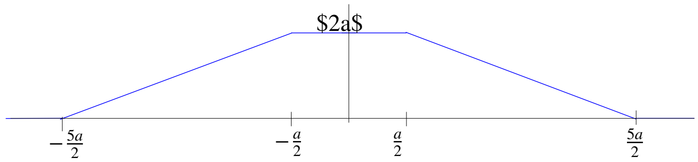
Case	Interval	u
1	$x < -2a$	$u(x,t) = \int_{x-a}^{x+a} 0\ ds = 0$
2	$-2a < x$	$u(x,t) = \int_{x-a}^{x+a} 0\ ds = 0$
3	$0 < x < 0 \rightarrow\leftarrow$	
4	$0 < x < 2a$	$u(x,t) = \int_{x-a}^{a} 1\ ds = 2a - x$
5	$-2a < x < 0$	$u(x,t) = \int_{-a}^{x+a} 1\ ds = x + 2a$
6	$x < 0, 0 < x \rightarrow\leftarrow$	



$t = \frac{3a}{2c}$

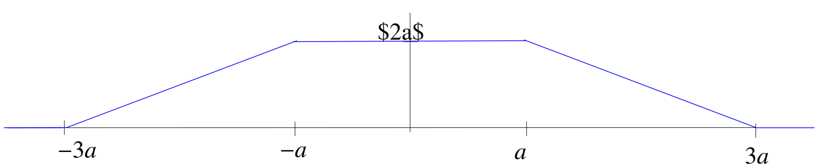
Case	Interval	u
1	$x < -\frac{5a}{2}$	$u(x,t) = \int_{x-\frac{3a}{2}}^{x+\frac{3a}{2}} 0\ ds = 0$
2	$\frac{5a}{2} < x$	$u(x,t) = \int_{x-\frac{3a}{2}}^{x+\frac{3a}{2}} 0\ ds = 0$
3	$x > \frac{a}{2}, x < -\frac{a}{2} \rightarrow\leftarrow$	
4	$\frac{a}{2} < x < \frac{5a}{2}$	$u(x,t) = \int_{x-\frac{3a}{2}}^{a} 1\ ds = -x + \frac{5a}{2}$
5	$-\frac{5a}{2} < x < -\frac{a}{2}$	$u(x,t) = \int_{-a}^{x+\frac{3a}{2}} 1\ ds = x + \frac{5a}{2}$
6	$-\frac{a}{2} < x < \frac{a}{2}$	$u(x,t) = \int_{-a}^{a} 1\ ds = 2a$

$t = \frac{2a}{c}$

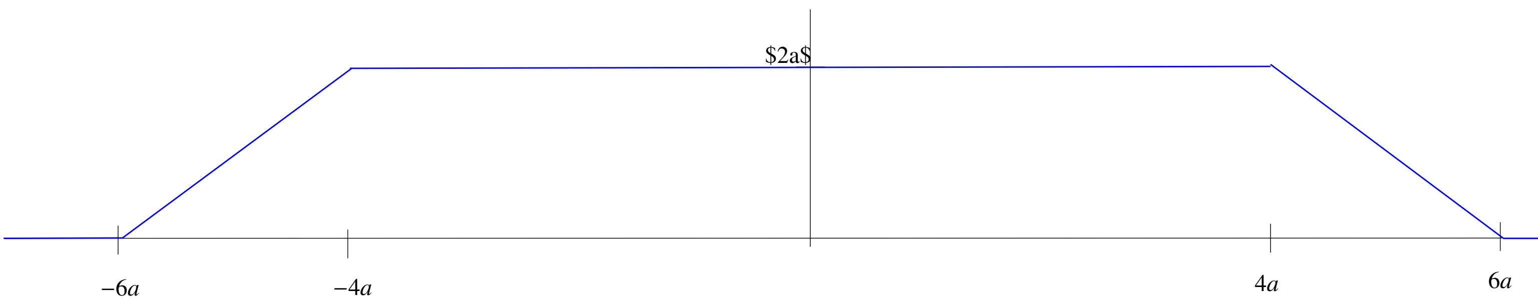


Case	Interval	u
1	$x < -3a$	$u(x,t) = \int_{x-2a}^{x+2a} 0\ ds = 0$
2	$-3a < x$	$u(x,t) = \int_{x-2a}^{x+2a} 0\ ds = 0$
3	$a < x, x < -a \rightarrow\leftarrow$	$u(x,t) = \int_{x-2a}^{x+2a} 1\ ds = 4a$
4	$a < x < 3a$	$u(x,t) = \int_{x-2a}^{a} 1\ ds = 3a - x$
5	$-3a < x < -a$	$u(x,t) = \int_{-a}^{x+2a} 1\ ds = x + 3a$
6	$-a < x < a$	$u(x,t) = \int_{-a}^{a} 1\ ds = 2a$

$t = \frac{5a}{c}$



Case	Interval	u
1	$x < -6a$	$u(x,t) = \int_{x-5a}^{x+5a} 0\, ds = 0$
2	$-6a < x$	$u(x,t) = \int_{x-5a}^{x+5a} 0\, ds = 0$
3	$4a < x, x < -4a \rightarrow\leftarrow$	
4	$4a < x < 6a$	$u(x,t) = \int_{x-5a}^{a} 1\, ds = 6a - x$
5	$-6a < x < -4a$	$u(x,t) = \int_{-a}^{x+5a} 1\, ds = x + 6a$
6	$-4a < x < 4a$	$u(x,t) = \int_{-a}^{a} 1\, ds = 2a$



pg 38 #9 Solve

$$\begin{aligned} u_{xx} - 3u_{xt} - 4u_{tt} &= 0 \\ u(x,0) &= x^2 \\ u_t(x,0) &= e^x \end{aligned}$$

First, we can rewrite the equation as

$$D^2u - 3DTu - 4T^2u = 0,$$

where $D = \frac{\partial}{\partial x}$ and $T = \frac{\partial}{\partial t}$. Factoring gives

$$(D + T)(D - 4T)u = 0.$$

Now, let's set $v = (D - 4T)u$. This gives the system of PDEs:

$$\begin{aligned} u_x - 4u_t &= v \\ v_x + v_t &= 0. \end{aligned}$$

To solve this system, we solve the second equation for v and then solve the first with v plugged back in. To solve the second equation, we recognize that it can be written as a dot product:

$$\nabla v \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.$$

Thus, ∇v is perpendicular to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, v is constant on lines parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. That is, v is constant on lines of the form $x - t = c$. So $v = f(x - t)$. Putting this into the first equation in our system gives

$$u_x - 4u_t = f(x - t).$$

Now we can do a change of coordinates to solve this equation.

$$\text{Let } \tilde{x} = x - 4t \text{ and } \tilde{t} = 4x + t.$$

Doing this change of coordinates gives us that

$$\begin{aligned} u_x &= u_{\tilde{x}} + 4u_{\tilde{t}} \\ u_t &= -4u_{\tilde{x}} + u_{\tilde{t}}. \end{aligned}$$

Rewriting the equation in terms of $\tilde{x}$ and $\tilde{t}$, we have

$$\begin{aligned} u_{\tilde{x}} + 4u_{\tilde{t}} + 4(-4u_{\tilde{x}} + u_{\tilde{t}}) &= f(x - t) \\ -15u_{\tilde{x}} &= f(x - t). \end{aligned}$$

Now, we just solve for u by integrating. This gives us

$$-15u = \int f(x - t)d\tilde{x} + g(\tilde{t}).$$

If we assume f has antiderivative cF (for some appropriate constant), we can write

$$u(x, t) = F(x - t) + g(4x + t).$$

Now, we use the iniitial conditions to find F and g.

$$u(x, 0) = x^2 \quad \Rightarrow \quad F(x) + g(4x) = x^2$$

and

$$u_t(x, 0) = e^x \quad \Rightarrow \quad -F'(x) + g'(4x) = e^x.$$

If we differentiate the first equation, we get

$$F'(x) + 4g'(4x) = 2x.$$

Adding these two equations gives us

$$5g'(4x) = e^x + 2x.$$

Integrating gives

$$\frac{5}{4}g(4x) = e^x + x^2 \quad \Rightarrow \quad g(x) = \frac{4}{5}e^{\frac{x}{4}} + \frac{1}{20}x^2.$$

Now, we put this back into the first equation and we get

$$F(x) = x^2 - g(4x) = x^2 - \frac{4}{5}(e^x + x^2) = \frac{1}{5}x^2 - \frac{4}{5}e^x.$$

Thus,

$$u(x, t) = F(x - t) + g(4x - t) = \frac{1}{5}(x - t)^2 - \frac{4}{5}e^{(x-t)} + \frac{4}{5}e^{\frac{4x+t}{4}} + \frac{1}{20}(4x + t)^2$$

pg 38 #10 Solve

$$\begin{aligned} u_{xx} + u_{xt} - 20u_{tt} &= 0 \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}$$

As in the previous problem, we rewrite the PDE using operators as

$$(D^2 + DT - 20T^2)u = 0,$$

where D, T are defined as before. Factoring gives us the equation

$$(D - 4T)(D + 5T)u = 0.$$

Now, as before, we substitute

$$v = (D + 5T)u$$

to get the system of equations

$$\begin{aligned} u_x + 5u_t &= v \\ v_x - 4v_t &= 0. \end{aligned}$$

Again, we recognize the second equation as a dot product and rewrite it as

$$\nabla v \cdot \begin{pmatrix} 1 \\ -4 \end{pmatrix} = 0.$$

This tells us that since ∇v is perpendicular to $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$, v is constant on lines of the form $4x + t = c$. That is, $v(x,t) = f(4x + t)$. Now, we plug this into the first equation to get the equation

$$u_x + 5u_t = f(4x + t).$$

We can use the change of coordinates

$$\tilde{x} = x + 5t, \tilde{t} = 5x - t$$

to get

$$u_x = u_{\tilde{x}} + 5u_{\tilde{t}} \text{ and } u_t = 5u_{\tilde{x}} - u_{\tilde{t}}.$$

This transforms our PDE to

$$u_{\tilde{x}} + 5u_{\tilde{t}} + 5(5u_{\tilde{x}} - u_{\tilde{t}}) = f(4x + t) \quad \Rightarrow \quad 6u_{\tilde{x}} = f(4x + t).$$

Integrating gives (again, assuming cF is the antiderivative of f with the appropriate constant c)

$$u = F(4x + t) + g(\tilde{t}).$$

Thus, $u(x,t) = F(4x + t) + g(5x - t)$.
Now, we use the initial conditions:

$$u(x,0) = \phi(x) \quad \Rightarrow \quad F(4x) + g(5x) = \phi(x)$$

and

$$u_t(x,0) = \psi(x) \quad \Rightarrow \quad F'(4x) - g'(5x) = \psi(x).$$

Differentiating the first gives

$$4F'(x) + 5g'(x) = \phi'(x).$$

Adding this to 5 times the second equation gives

$$9F'(4x) = \phi'(x) + 5\psi(x).$$

Thus,

$$F'(x) = \frac{1}{9}\left(\phi'\left(\frac{x}{4}\right) + 5\psi\left(\frac{x}{4}\right)\right).$$

Integrating gives

$$F(x) = \frac{1}{36}\phi\left(\frac{x}{4}\right) + \frac{5}{36}\int_0^{x/4} \psi(s)\, ds.$$

Using the first equation again, we get

$$g(5x) = \phi(x) - F(4x) = \phi(x) - \frac{1}{36}\phi(x) - \frac{5}{36}\int_0^{x/4} \psi(s)\, ds$$

Thus

$$g(x) = \phi(x/5) - F(4x/5) = \phi(x/5) - \frac{1}{36}\phi(x/5) - \frac{5}{36}\int_0^{x/20} \psi(s)\, ds = \frac{35}{36}\phi(x) - \frac{5}{36}\int_0^{x/20} \psi(s)\, ds.$$

This gives us our solution:

$$u(x,t) = \frac{1}{36}\phi\left(\frac{4x+t}{4}\right) + \frac{5}{36}\int_0^{\frac{4x+t}{4}} \psi(s)\, ds + \frac{35}{36}\phi(5x-t) - \frac{5}{36}\int_0^{\frac{5x-t}{20}} \psi(s)\, ds.$$

1. In a Vector Calculus class, Green's Theorem is presented as:

$$\oint_{\partial D} \mathbf{F}\cdot\mathbf{n}\, ds = \int\int_D div\mathbf{F}(x,y)\, dA, \tag{1}$$

where **F** is a vector-valued function. Use this to prove these other versions of Green's Theorem.

(a)

$$\int\int_D f\Delta g\, dA = \oint_{\partial D} f\nabla g\cdot\mathbf{n}\, ds - \int\int_D \nabla f\cdot\nabla g\, dA.$$

To get this, we replace, in (1) $F = f\nabla g$ to get

$$\oint_{\partial D} f\nabla g\cdot\mathbf{n}\, ds = \int\int_D div\,(f\nabla g)\; dA.$$

Using the product rule, we compute

$$div(f\nabla g) = \nabla f\cdot\nabla g + f\Delta g.$$

Thus, we have

$$\oint_{\partial D} f\nabla g\cdot\mathbf{n}\, ds = \int\int_D \nabla f\cdot\nabla g + f\Delta g\, dA.$$

Rearranging gives the desired result.

(b)

$$\int\int_D (f\Delta g - g\Delta f)\, dA = \oint_{\partial D} (f\nabla g - g\nabla f) \cdot \mathbf{n}\, ds$$

For this one, we substitute in (1) $F = f\nabla g - g\nabla f$ to get

$$\oint_{\partial D} (f\nabla g - g\nabla f) \cdot \mathbf{n}\, ds = \int\int_D div(f\nabla g - g\nabla f)\, dA.$$

Using the product rule twice, we compute

$$div(f\nabla g - g\nabla f) = \nabla f \cdot \nabla g + f\Delta g - \nabla g \cdot \nabla f - g\Delta f = f\Delta g - g\Delta f.$$

Thus, we have

$$\oint_{\partial D} (f\nabla g - g\nabla f) \cdot \mathbf{n}\, ds = \int\int_D f\Delta g - g\Delta f\, dA, \tag{2}$$

which the desired result.