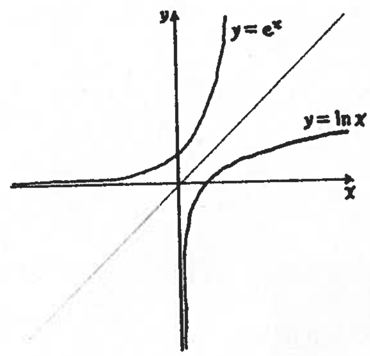
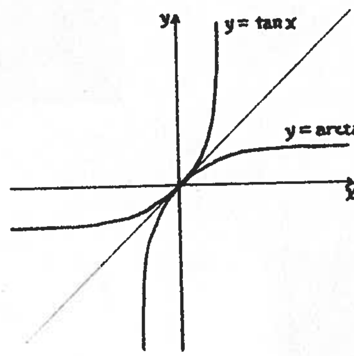
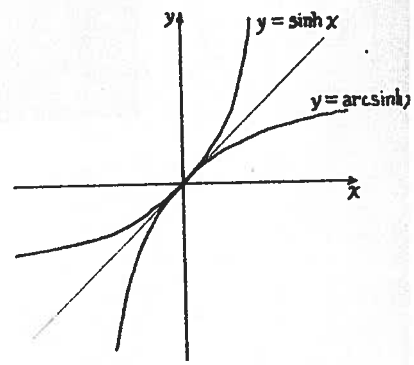
CHAPTER 7





The graphs of the functions of this chapter appear as reflections of each other—the natural exponential and logarithmic functions, the restricted tangent and inverse tangent functions, and the hyperbolic sine function and its inverse.



Inverse Functions:

Exponential, Logarithmic, and Inverse Trigonometric Functions

The common theme that links the functions of this chapter is that they occur as pairs of inverse functions. In particular, two of the most important functions that occur in mathematics and its applications are the exponential function $f(x) = a^x$ and its inverse function, the logarithmic function $g(x) = \log_a x$. In this chapter we investigate their properties, compute their derivatives, and use them to describe exponential growth and decay in biology, physics, chemistry, and other sciences. We also study the inverses of trigonometric and hyperbolic functions. Finally, we look at a method (l'Hospital's Rule) for computing difficult limits and apply it to sketching curves.

There are two possible ways of defining the exponential and logarithmic functions and developing their properties and derivatives. One is to start with the exponential function (defined as in algebra or precalculus courses) and then define the logarithm as its inverse. That is the approach taken in Sections 7.2, 7.3, and 7.4 and is probably the most intuitive method. The other way is to start by defining the logarithm as an integral and then define the exponential function as its inverse. This approach is followed in Sections 7.2*, 7.3*, and 7.4* and, although it is less intuitive, many instructors prefer it because it is more rigorous and the properties follow more easily. You need only read one of these two approaches (whichever your instructor recommends).

7.1 Inverse Functions

Table 1 gives data from an experiment in which a bacteria culture started with 100 bacteria in a limited nutrient medium; the size of the bacteria population was recorded at hourly intervals. The number of bacteria N is a function of the time t: $N = f(t)$.

Suppose, however, that the biologist changes her point of view and becomes interested in the time required for the population to reach various levels. In other words, she is thinking of t as a function of N. This function is called the *inverse function* of f, denoted by f^{-1}, and read "f inverse." Thus, $t = f^{-1}(N)$ is the time required for the population level to reach N. The values of f^{-1} can be found by reading Table 1 from right to left or by consulting Table 2. For instance, $f^{-1}(550) = 6$ because $f(6) = 550$.

TABLE 1 N as a function of t

t (hours)	$N = f(t)$ = population at time t
0	100
1	168
2	259
3	358
4	445
5	509
6	550
7	573
8	586

TABLE 2 t as a function of N

N	$t = f^{-1}(N)$ = time to reach N bacteria
100	0
168	1
259	2
358	3
445	4
509	5
550	6
573	7
586	8

Not all functions possess inverses. Let's compare the functions f and g whose arrow diagrams are shown in Figure 1.

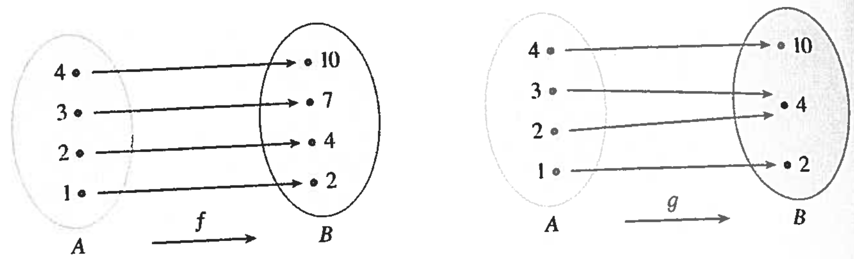


FIGURE 1

Note that f never takes on the same value twice (any two inputs in A have different outputs), whereas g does take on the same value twice (both 2 and 3 have the same output, 4). In symbols,

$$g(2) = g(3)$$

but

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Functions that have this property are called *one-to-one functions*.

IIII In the language of inputs and outputs, this definition says that f is one-to-one if each output corresponds to only one input.

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

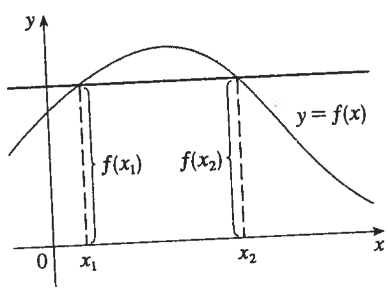


FIGURE 2
This function is not one-to-one because $f(x_1) = f(x_2)$.

If a horizontal line intersects the graph of f in more than one point, then we see from Figure 2 that there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. This means that f is not one-to-one. Therefore, we have the following geometric method for determining whether a function is one-to-one.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

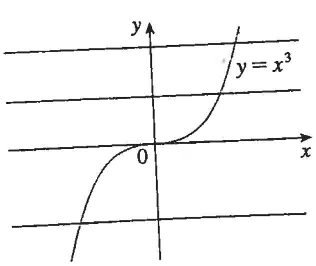


FIGURE 3
$f(x) = x^3$ is one-to-one.

EXAMPLE 1 Is the function $f(x) = x^3$ one-to-one?

SOLUTION 1 If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers can't have the same cube). Therefore, by Definition 1, $f(x) = x^3$ is one-to-one.

SOLUTION 2 From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one.

EXAMPLE 2 Is the function $g(x) = x^2$ one-to-one?

SOLUTION 1 This function is not one-to-one because, for instance,

$$g(1) = 1 = g(-1)$$

and so 1 and -1 have the same output.

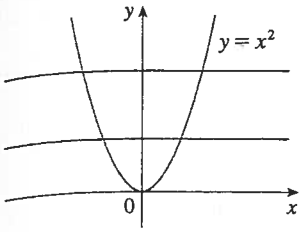


FIGURE 4
$g(x) = x^2$ is not one-to-one.

SOLUTION 2 From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one.

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

> **2 Definition** Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by
>
> $$f^{-1}(y) = x \iff f(x) = y$$
>
> for any y in B.

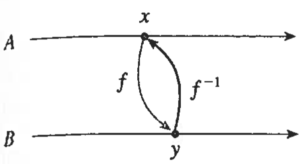


FIGURE 5

This definition says that if f maps x into y, then f^{-1} maps y back into x. (If f were not one-to-one, then f^{-1} would not be uniquely defined.) The arrow diagram in Figure 5 indicates that f^{-1} reverses the effect of f. Note that

$$\text{domain of } f^{-1} = \text{range of } f$$
$$\text{range of } f^{-1} = \text{domain of } f$$

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$, then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

CAUTION ◦ Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \quad \text{does } \textit{not} \text{ mean} \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ could, however, be written as $[f(x)]^{-1}$.

EXAMPLE 3 If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

SOLUTION From the definition of f^{-1} we have

$$f^{-1}(7) = 3 \quad \text{because} \quad f(3) = 7$$
$$f^{-1}(5) = 1 \quad \text{because} \quad f(1) = 5$$
$$f^{-1}(-10) = 8 \quad \text{because} \quad f(8) = -10$$

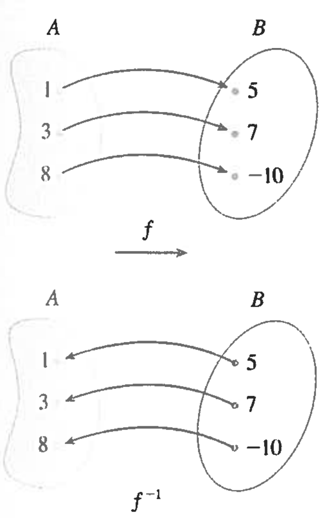


FIGURE 6
The inverse function reverses inputs and outputs.

The diagram in Figure 6 makes it clear how f^{-1} reverses the effect of f in this case.

The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f, we usually reverse the roles of x and y in Definition 2 and write

3
$$f^{-1}(x) = y \iff f(y) = x$$

By substituting for y in Definition 2 and substituting for x in (3), we get the following **cancellation equations**:

4
$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

The first cancellation equation says that if we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started (see the machine diagram in Figure 7). Thus, f^{-1} undoes what f does. The second equation says that f undoes what f^{-1} does.

$$x \longrightarrow f \longrightarrow f(x) \longrightarrow f^{-1} \longrightarrow x$$

FIGURE 7

For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$ and the cancellation equations become

$$f^{-1}(f(x)) = (x^3)^{1/3} = x$$
$$f(f^{-1}(x)) = (x^{1/3})^3 = x$$

These equations simply say that the cube function and the cube root function cancel each other when applied in succession.

Now let's see how to compute inverse functions. If we have a function $y = f(x)$ and are able to solve this equation for x in terms of y, then according to Definition 2 we must have $x = f^{-1}(y)$. If we want to call the independent variable x, we then interchange x and y and arrive at the equation $y = f^{-1}(x)$.

5 How to Find the Inverse Function of a One-To-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

EXAMPLE 4 Find the inverse function of $f(x) = x^3 + 2$.

SOLUTION According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for x:

$$x^3 = y - 2$$
$$x = \sqrt[3]{y - 2}$$

|||| In Example 4, notice how f^{-1} reverses the effect of f. The function f is the rule "Cube, then add 2"; f^{-1} is the rule "Subtract 2, then take the cube root."

Finally, we interchange x and y:

$$y = \sqrt[3]{x - 2}$$

Therefore, the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$.

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f. Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$. (See Figure 8.)

Therefore, as illustrated by Figure 9:

> The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

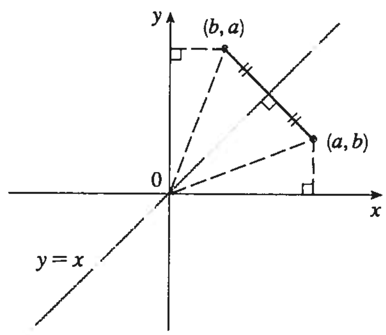


FIGURE 8

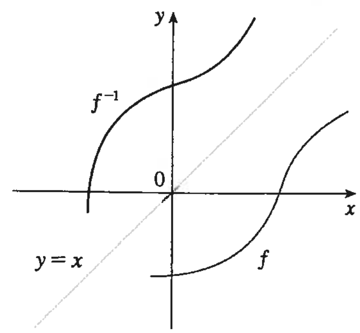


FIGURE 9

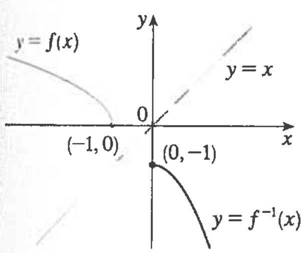


FIGURE 10

EXAMPLE 5 Sketch the graphs of $f(x) = \sqrt{-1 - x}$ and its inverse function using the same coordinate axes.

SOLUTION First we sketch the curve $y = \sqrt{-1 - x}$ (the top half of the parabola $y^2 = -1 - x$, or $x = -y^2 - 1$) and then we reflect about the line $y = x$ to get the graph of f^{-1}. (See Figure 10.) As a check on our graph, notice that the expression for f^{-1} is $f^{-1}(x) = -x^2 - 1$, $x \geq 0$. So the graph of f^{-1} is the right half of the parabola $y = -x^2 - 1$ and this seems reasonable from Figure 10.

|||| The Calculus of Inverse Functions

Now let's look at inverse functions from the point of view of calculus. Suppose that f is both one-to-one and continuous. We think of a continuous function as one whose graph has no break in it. (It consists of just one piece.) Since the graph of f^{-1} is obtained from the graph of f by reflecting about the line $y = x$, the graph of f^{-1} has no break in it either (see Figure 9). Thus, we might expect that f^{-1} is also a continuous function.

This geometrical argument does not prove the following theorem but at least it makes the theorem plausible. A proof can be found in Appendix F.

> **6 Theorem** If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

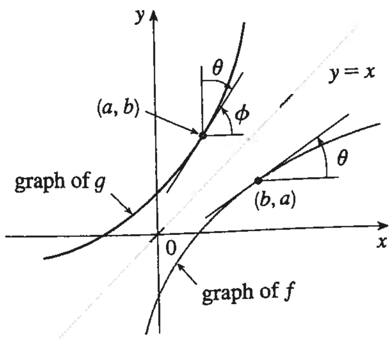


FIGURE 11

Now suppose that f is a one-to-one differentiable function. Geometrically we can think of a differentiable function as one whose graph has no corner or kink in it. We get the graph of f^{-1} by reflecting the graph of f about the line $y = x$, so the graph of f^{-1} has no corner or kink in it either. We therefore expect that f^{-1} is also differentiable (except where its tangents are vertical). In fact, we can predict the value of the derivative of f^{-1} at a given point by a geometric argument. In Figure 11 the graphs of f and its inverse $g = f^{-1}$ are shown. If $f(b) = a$, then $g(a) = f^{-1}(a) = b$ and $g'(a)$ is the slope of the tangent to the graph of g at (a, b), which is $\tan \phi$. Likewise, $f'(b) = \tan \theta$. From Figure 11 we see that $\theta + \phi = \pi/2$, so

$$g'(a) = \tan \phi = \tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan \theta} = \frac{1}{f'(b)}$$

that is,

$$g'(a) = \frac{1}{f'(g(a))}$$

7 Theorem If f is a one-to-one differentiable function with inverse function $g = f^{-1}$ and $f'(g(a)) \neq 0$, then the inverse function is differentiable at a and

$$g'(a) = \frac{1}{f'(g(a))}$$

Proof Write the definition of derivative as in Equation 3.1.3:

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

By (3) we have

$$g(x) = y \iff f(y) = x$$

and

$$g(a) = b \iff f(b) = a$$

Since f is differentiable, it is continuous, so $g = f^{-1}$ is continuous by Theorem 6. Thus, if $x \to a$, then $g(x) \to g(a)$, that is, $y \to b$. Therefore

$$\begin{aligned} g'(a) &= \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{y \to b} \frac{y - b}{f(y) - f(b)} \\ &= \lim_{y \to b} \frac{1}{\dfrac{f(y) - f(b)}{y - b}} = \frac{1}{\displaystyle\lim_{y \to b} \frac{f(y) - f(b)}{y - b}} \\ &= \frac{1}{f'(b)} = \frac{1}{f'(g(a))} \end{aligned}$$

NOTE 1 ▫ Replacing a by the general number x in the formula of Theorem 7, we get

$$\boxed{8} \qquad g'(x) = \frac{1}{f'(g(x))}$$

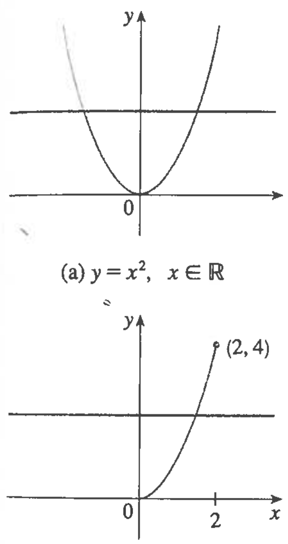
If we write $y = g(x)$, then $f(y) = x$, so Equation 8, when expressed in Leibniz notation, becomes

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

NOTE 2 ▫ If it is known in advance that f^{-1} is differentiable, then its derivative can be computed more easily than in the proof of Theorem 7 by using implicit differentiation. If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating the equation $f(y) = x$ implicitly with respect to x, remembering that y is a function of x, and using the Chain Rule, we get

$$f'(y)\,\frac{dy}{dx} = 1$$

Therefore
$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{\dfrac{dx}{dy}}$$



(a) $y = x^2,\ x \in \mathbb{R}$

(b) $f(x) = x^2,\ 0 \leqslant x \leqslant 2$

FIGURE 12

EXAMPLE 6 Although the function $y = x^2$, $x \in \mathbb{R}$, is not one-to-one and therefore does not have an inverse function, we can turn it into a one-to-one function by restricting its domain. For instance, the function $f(x) = x^2$, $0 \leqslant x \leqslant 2$, is one-to-one (by the Horizontal Line Test) and has domain $[0, 2]$ and range $[0, 4]$. (See Figure 12.) Thus, f has an inverse function $g = f^{-1}$ with domain $[0, 4]$ and range $[0, 2]$.

Without computing a formula for g' we can still calculate $g'(1)$. Since $f(1) = 1$, we have $g(1) = 1$. Also $f'(x) = 2x$. So by Theorem 7 we have

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(1)} = \frac{1}{2}$$

In this case it is easy to find g explicitly. In fact, $g(x) = \sqrt{x}$, $0 \leqslant x \leqslant 4$. [In general, we could use the method given by (5).] Then $g'(x) = 1/(2\sqrt{x})$, so $g'(1) = \frac{1}{2}$, which agrees with the preceding computation. The functions f and g are graphed in Figure 13.

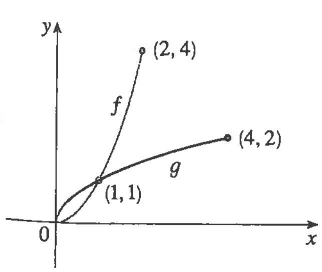


FIGURE 13

EXAMPLE 7 If $f(x) = 2x + \cos x$, find $(f^{-1})'(1)$.+

SOLUTION Notice that f is one-to-one because

$$f'(x) = 2 - \sin x > 0$$

and so f is increasing. To use Theorem 7 we need to know $f^{-1}(1)$ and we can find it by inspection:

$$f(0) = 1 \quad \Rightarrow \quad f^{-1}(1) = 0$$

Therefore
$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2 - \sin 0} = \frac{1}{2}$$

7.1 Exercises

1. (a) What is a one-to-one function?
(b) How can you tell from the graph of a function whether it is one-to-one?

2. (a) Suppose f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What is the domain of f^{-1}? What is the range of f^{-1}?
(b) If you are given a formula for f, how do you find a formula for f^{-1}?
(c) If you are given the graph of f, how do you find the graph of f^{-1}?

3–16 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

3.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

4.

x	1	2	3	4	5	6
$f(x)$	1	2	4	8	16	32

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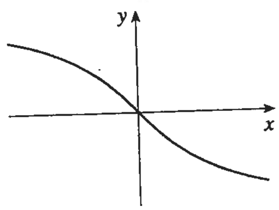

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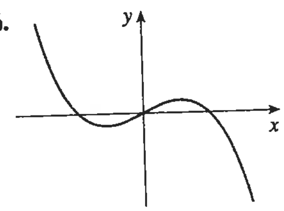

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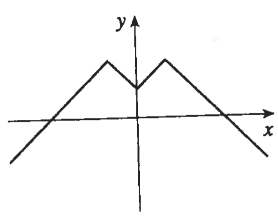

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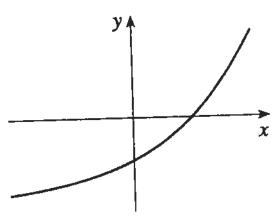


9. $f(x) = \frac{1}{2}(x + 5)$

10. $f(x) = 1 + 4x - x^2$

11. $g(x) = \sqrt{x}$

12. $g(x) = |x|$

13. $h(x) = x^4 + 5$

14. $h(x) = x^4 + 5, \quad 0 \leqslant x \leqslant 2$

15. $f(t)$ is the height of a football t seconds after kickoff.

16. $f(t)$ is your height at age t.

17–18 Use a graph to decide whether f is one-to-one.

17. $f(x) = x^3 - x$

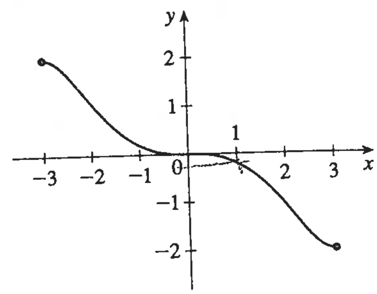
18. $f(x) = x^3 + x$

19. If f is a one-to-one function such that $f(2) = 9$, what is $f^{-1}(9)$?

20. If $f(x) = x + \cos x$, find $f^{-1}(1)$.

21. If $h(x) = x + \sqrt{x}$, find $h^{-1}(6)$.

22. The graph of f is given.
(a) Why is f one-to-one?
(b) State the domain and range of f^{-1}.
(c) Estimate the value of $f^{-1}(1)$.



23. The formula $C = \frac{5}{9}(F - 32)$, where $F \geqslant -459.67$, expresses the Celsius temperature C as a function of the Fahrenheit temperature F. Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

24. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

25–30 Find a formula for the inverse of the function.

25. $f(x) = 3 - 2x$

26. $f(x) = \dfrac{4x - 1}{2x + 3}$

27. $f(x) = \sqrt{10 - 3x}$

28. $y = 2x^3 + 3$

29. $y = \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$

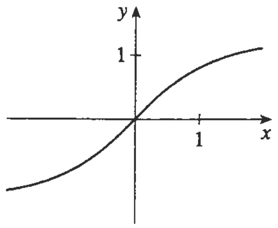
30. $f(x) = 2x^2 - 8x, \quad x \geqslant 2$

31–32 Find an explicit formula for f^{-1} and use it to graph f^{-1}, f, and the line $y = x$ on the same screen. To check your work, see whether the graphs of f and f^{-1} are reflections about the line.

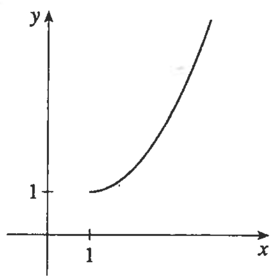
31. $f(x) = 1 - 2/x^2, \quad x > 0$

32. $f(x) = \sqrt{x^2 + 2x}, \quad x > 0$

33. Use the given graph of f to sketch the graph of f^{-1}.



34. Use the given graph of f to sketch the graphs of f^{-1} and $1/f$.



35–38 ||||

(a) Show that f is one-to-one.
(b) Use Theorem 7 to find $g'(a)$, where $g = f^{-1}$.
(c) Calculate $g(x)$ and state the domain and range of g.
(d) Calculate $g'(a)$ from the formula in part (c) and check that it agrees with the result of part (b).
(e) Sketch the graphs of f and g on the same axes.

35. $f(x) = x^3, \quad a = 8$

36. $f(x) = \sqrt{x - 2}, \quad a = 2$

37. $f(x) = 9 - x^2, \quad 0 \leqslant x \leqslant 3, \quad a = 8$

38. $f(x) = 1/(x - 1), \quad x > 1, \quad a = 2$

39–42 |||| Find $(f^{-1})'(a)$.

39. $f(x) = x^3 + x + 1, \quad a = 1$

40. $f(x) = x^5 - x^3 + 2x, \quad a = 2$

41. $f(x) = 3 + x^2 + \tan(\pi x/2), \quad -1 < x < 1, \quad a = 3$

42. $f(x) = \sqrt{x^3 + x^2 + x + 1}, \quad a = 2$

43. Suppose g is the inverse function of f and $f(4) = 5, f'(4) = \frac{2}{3}$. Find $g'(5)$.

44. Suppose g is the inverse function of a differentiable function f and let $G(x) = 1/g(x)$. If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, find $G'(2)$.

CAS **45.** Use a computer algebra system to find an explicit expression for the inverse of the function $f(x) = \sqrt{x^3 + x^2 + x + 1}$. (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)

46. Show that $h(x) = \sin x, x \in \mathbb{R}$, is not one-to-one, but its restriction $f(x) = \sin x, -\pi/2 \leqslant x \leqslant \pi/2$, is one-to-one. Compute the derivative of $f^{-1} = \sin^{-1}$ by the method of Note 2.

47. (a) If we shift a curve to the left, what happens to its reflection about the line $y = x$? In view of this geometric principle, find an expression for the inverse of $g(x) = f(x + c)$, where f is a one-to-one function.
(b) Find an expression for the inverse of $h(x) = f(cx)$, where $c \neq 0$.

48. (a) If f is a one-to-one, twice differentiable function with inverse function g, show that

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}$$

(b) Deduce that if f is increasing and concave upward, then its inverse function is concave downward.

|||| 7.2 Exponential Functions and Their Derivatives

|||| If your instructor has assigned Sections 7.2*, 7.3*, and 7.4*, you don't need to read Sections 7.2–7.4 (pp. 421–450).

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x, is the exponent. It should not be confused with the power function $g(x) = x^2$, in which the variable is the base.

In general, an **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant. Let's recall what this means.

If $x = n$, a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

If $x = 0$, then $a^0 = 1$, and if $x = -n$, where n is a positive integer, then

$$a^{-n} = \frac{1}{a^n}$$

If x is a rational number, $x = p/q$, where p and q are integers and $q > 0$, then

$$a^x = a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

But what is the meaning of a^x if x is an irrational number? For instance, what is mea $2^{\sqrt{3}}$ or 5^{π}?

To help us answer this question we first look at the graph of the function $y = 2^x$, v x is rational. A representation of this graph is shown in Figure 1. We want to enlarg domain of $y = 2^x$ to include both rational and irrational numbers.

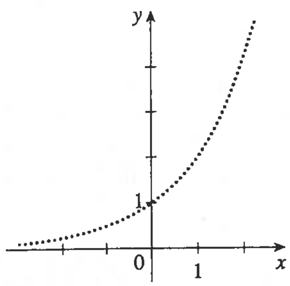


FIGURE 1
Representation of $y = 2^x$, x rational

There are holes in the graph in Figure 1 corresponding to irrational values of x. We to fill in the holes by defining $f(x) = 2^x$, where $x \in \mathbb{R}$, so that f is an increasing fun In particular, since the irrational number $\sqrt{3}$ satisfies

$$1.7 < \sqrt{3} < 1.8$$

we must have

$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$

and we know what $2^{1.7}$ and $2^{1.8}$ mean because 1.7 and 1.8 are rational numbers. Simi if we use better approximations for $\sqrt{3}$, we obtain better approximations for $2^{\sqrt{3}}$:

$$\begin{array}{rcl} 1.73 < \sqrt{3} < 1.74 & \Rightarrow & 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\ 1.732 < \sqrt{3} < 1.733 & \Rightarrow & 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\ 1.7320 < \sqrt{3} < 1.7321 & \Rightarrow & 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\ 1.73205 < \sqrt{3} < 1.73206 & \Rightarrow & 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206} \\ \vdots \quad \vdots & & \vdots \quad \vdots \end{array}$$

A proof of this fact is given in J. Marsden and A. Weinstein, *Calculus Unlimited* (Menlo Park, CA: Benjamin/Cummings, 1980).

It can be shown that there is exactly one number that is greater than all of the nur

$$2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \quad 2^{1.7320}, \quad 2^{1.73205}, \quad \ldots$$

and less than all of the numbers

$$2^{1.8}, \quad 2^{1.74}, \quad 2^{1.733}, \quad 2^{1.7321}, \quad 2^{1.73206}, \quad \ldots$$

We define $2^{\sqrt{3}}$ to be this number. Using the preceding approximation process we can pute it correct to six decimal places:

$$2^{\sqrt{3}} \approx 3.321997$$

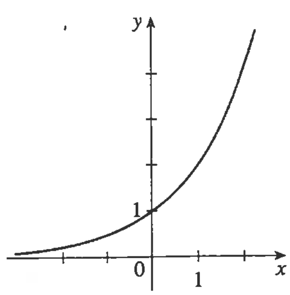


FIGURE 2
$y = 2^x$, x real

Similarly, we can define 2^x where x is any irrational number. Figure 2 shows how a holes in Figure 1 have been filled to complete the graph of the function $f(x) = 2^x$, x

In general, if a is any positive number, we define

1 $$a^x = \lim_{r \to x} a^r \qquad r \text{ rational}$$

This definition makes sense because any irrational number can be approximated as closely as we like by a rational number. For instance, because $\sqrt{3}$ has the decimal representation $\sqrt{3} = 1.7320508\ldots$, Definition 1 says that $2^{\sqrt{3}}$ is the limit of the sequence of numbers

$$2^{1.7},\quad 2^{1.73},\quad 2^{1.732},\quad 2^{1.7320},\quad 2^{1.73205},\quad 2^{1.732050},\quad 2^{1.7320508},\quad \ldots$$

Similarly, 5^{π} is the limit of the sequence of numbers

$$5^{3.1},\quad 5^{3.14},\quad 5^{3.141},\quad 5^{3.1415},\quad 5^{3.14159},\quad 5^{3.141592},\quad 5^{3.1415926},\quad \ldots$$

It can be shown that Definition 1 uniquely specifies a^x and makes the function $f(x) = a^x$ continuous.

The graphs of members of the family of functions $y = a^x$ are shown in Figure 3 for various values of the base a. Notice that all of these graphs pass through the same point $(0, 1)$ because $a^0 = 1$ for $a \neq 0$. Notice also that as the base a gets larger, the exponential function grows more rapidly (for $x > 0$).

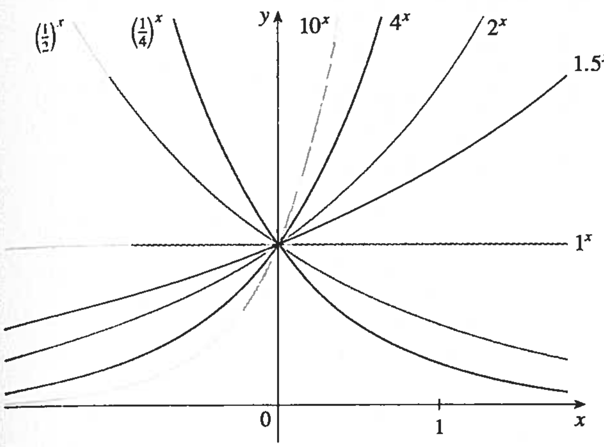


FIGURE 3
Members of the family of exponential functions

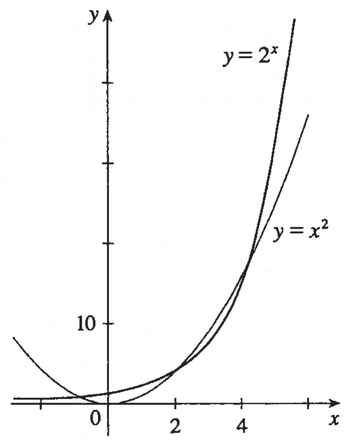


FIGURE 4

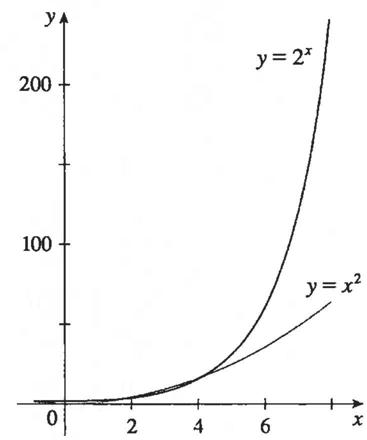
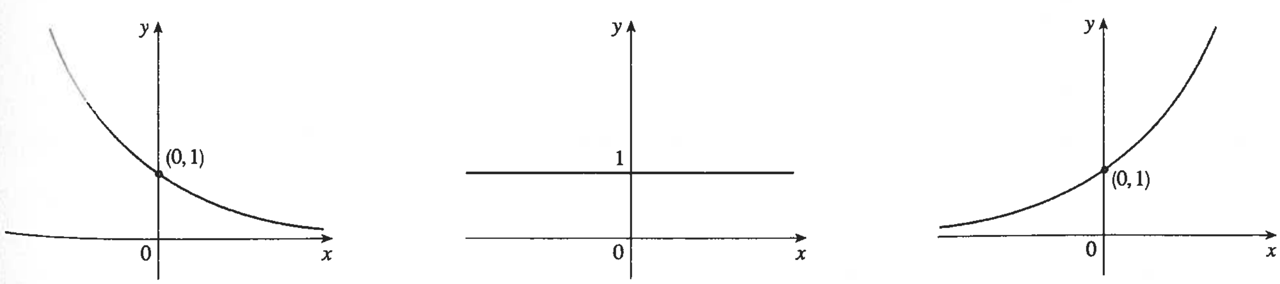


FIGURE 5

Figure 4 shows how the exponential function $y = 2^x$ compares with the power function $y = x^2$. The graphs intersect three times, but ultimately the exponential curve $y = 2^x$ grows far more rapidly than the parabola $y = x^2$. (See also Figure 5.)

You can see from Figure 3 that there are basically three kinds of exponential functions $y = a^x$. If $0 < a < 1$, the exponential function decreases; if $a = 1$, it is a constant; and if $a > 1$, it increases. These three cases are illustrated in Figure 6. Since $(1/a)^x = 1/a^x = a^{-x}$, the graph of $y = (1/a)^x$ is just the reflection of the graph of $y = a^x$ about the y-axis.



(a) $y = a^x$, $0 < a < 1$

(b) $y = 1^x$

(c) $y = a^x$, $a > 1$

FIGURE 6

The properties of the exponential function are summarized in the following theorem.

[2] Theorem If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. In particular, $a^x > 0$ for all x. If $0 < a < 1$, $f(x) = a^x$ is a decreasing function; if $a > 1$, f is an increasing function. If $a, b > 0$ and $x, y \in \mathbb{R}$, then

1. $a^{x+y} = a^x a^y$ **2.** $a^{x-y} = \dfrac{a^x}{a^y}$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$

The reason for the importance of the exponential function lies in properties 1–4, which are called the **Laws of Exponents**. If x and y are rational numbers, then these laws are well known from elementary algebra. For arbitrary real numbers x and y these laws can be deduced from the special case where the exponents are rational by using Equation 1.

The following limits can be read from the graphs shown in Figure 6 or proved from the definition of a limit at infinity. (See Exercise 73 in Section 7.3.)

[3] If $a > 1$, then $\lim_{x\to\infty} a^x = \infty$ and $\lim_{x\to-\infty} a^x = 0$

If $0 < a < 1$, then $\lim_{x\to\infty} a^x = 0$ and $\lim_{x\to-\infty} a^x = \infty$

In particular, if $a \neq 1$, then the x-axis is a horizontal asymptote of the graph of the exponential function $y = a^x$.

EXAMPLE 1

(a) Find $\lim_{x\to\infty} (2^{-x} - 1)$.

(b) Sketch the graph of the function $y = 2^{-x} - 1$.

SOLUTION

(a)
$$\begin{aligned}\lim_{x\to\infty} (2^{-x} - 1) &= \lim_{x\to\infty} \left[\left(\tfrac{1}{2}\right)^x - 1\right] \\ &= 0 - 1 \qquad \left[\text{by (3) with } a = \tfrac{1}{2} < 1\right] \\ &= -1\end{aligned}$$

(b) We write $y = \left(\frac{1}{2}\right)^x - 1$ as in part (a). The graph of $y = \left(\frac{1}{2}\right)^x$ is shown in Figure 3, so we shift it down one unit to obtain the graph of $y = \left(\frac{1}{2}\right)^x - 1$ shown in Figure 7. (For a review of shifting graphs, see Section 1.3.) Part (a) shows that the line $y = -1$ is a horizontal asymptote.

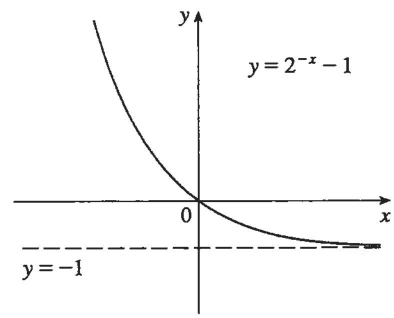


FIGURE 7

Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of population growth and radioactive decay. In Chapter 10 we will pursue these and other applications in greater detail.

In Section 3.4 we considered a bacteria population that doubles every hour and saw that if the initial population is n_0, then the population after t hours is given by the function $f(t) = n_0 2^t$. This population function is a constant multiple of the exponential function $y = 2^t$, so it exhibits the rapid growth that we observed in Figures 2 and 5. Under ideal conditions (unlimited space and nutrition and freedom from disease) this exponential growth is typical of what actually occurs in nature.

What about the human population? Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot.

TABLE 1

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

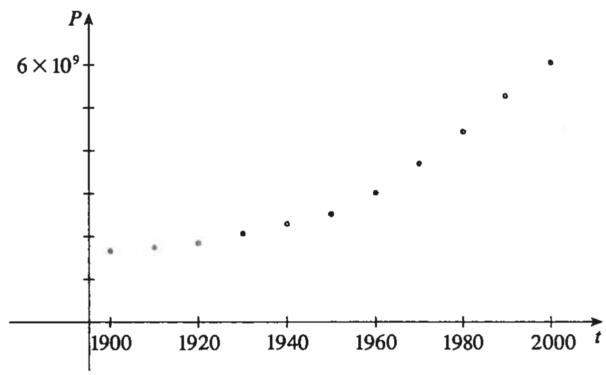


FIGURE 8 Scatter plot for world population growth

The pattern of the data points in Figure 8 suggests exponential growth, so we use a graphing calculator with exponential regression capability to apply the method of least squares and obtain the exponential model

$$P = (0.008079266) \cdot (1.013731)^t$$

Figure 9 shows the graph of this exponential function together with the original data points. We see that the exponential curve fits the data reasonably well. The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

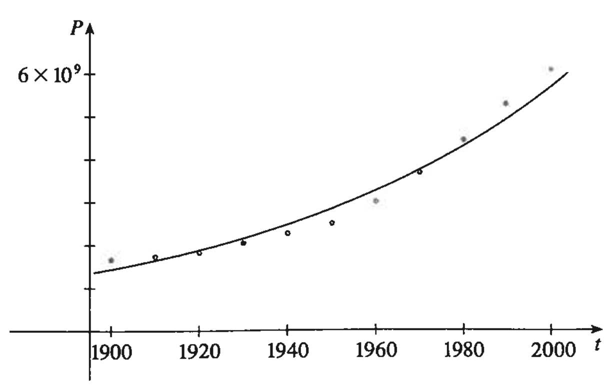


FIGURE 9 Exponential model for population growth

Exponential functions also occur in the study of the decay of radioactive substances. For instance, when physicists say the *half-life* of strontium-90, ^{90}Sr, is 25 years they mean that half of any given quantity of ^{90}Sr will disintegrate in 25 years. So if the initial mass of a sample of ^{90}Sr is 24 mg and the mass that remains after t years is $m(t)$, then

$$m(25) = \frac{1}{2}(24) \qquad m(50) = \frac{1}{2^2}(24)$$

$$m(75) = \frac{1}{2^3}(24) \qquad m(100) = \frac{1}{2^4}(24)$$

From this pattern, we see that the mass remaining after t years is

4 $$m(t) = \frac{1}{2^{t/25}}(24) = 24 \cdot 2^{-t/25}$$

This is an exponential function with base $a = 2^{-1/25}$. (See Exercise 55 and Section 10.4.)

Derivatives of Exponential Functions

Let's try to compute the derivative of the exponential function $f(x) = a^x$ using the definition of a derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

The factor a^x doesn't depend on h, so we can take it in front of the limit:

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

Notice that the limit is the value of the derivative of f at 0, that is,

$$\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$$

Therefore, we have shown that if the exponential function $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere and

5 $$f'(x) = f'(0)a^x$$

This equation says that *the rate of change of any exponential function is proportional to the function itself.* (The slope is proportional to the height.)

Numerical evidence for the existence of $f'(0)$ is given in the table at the left for the cases $a = 2$ and $a = 3$. (Values are stated correct to four decimal places. For the case $a = 2$, see also Example 3 in Section 3.1.) It appears that the limits exist and

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

$$\text{for } a = 2, \quad f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.69$$

$$\text{for } a = 3, \quad f'(0) = \lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.10$$

In fact, it can be proved that these limits exist and, correct to six decimal places, the values are

6 $$\frac{d}{dx}(2^x)\bigg|_{x=0} \approx 0.693147 \qquad \frac{d}{dx}(3^x)\bigg|_{x=0} \approx 1.098612$$

Thus, from Equation 5 we have

7 $$\frac{d}{dx}(2^x) \approx (0.69)2^x \qquad \frac{d}{dx}(3^x) \approx (1.10)3^x$$

Of all possible choices for the base a in Equation 5, the simplest differentiation formula occurs when $f'(0) = 1$. In view of the estimates of $f'(0)$ for $a = 2$ and $a = 3$, it seems reasonable that there is a number a between 2 and 3 for which $f'(0) = 1$. It is traditional to denote this value by the letter e. Thus, we have the following definition.

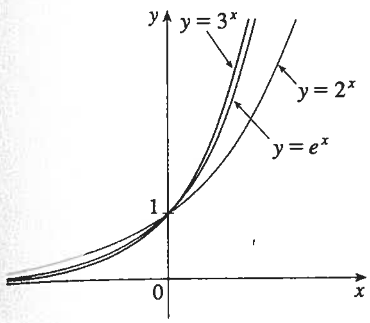


FIGURE 10

8 **Definition of the Number e**

e is the number such that $$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

Geometrically, this means that of all the possible exponential functions $y = a^x$, the function $f(x) = e^x$ is the one whose tangent line at $(0, 1)$ has a slope $f'(0)$ that is exactly 1 (see Figures 10 and 11).

If we put $a = e$ and, therefore, $f'(0) = 1$ in Equation 5, it becomes the following important differentiation formula.

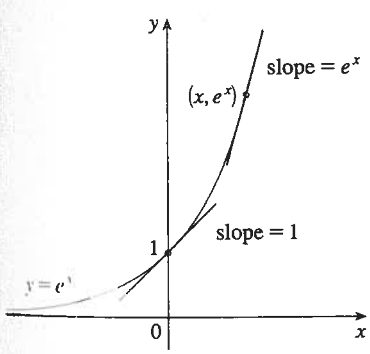


FIGURE 11

9 **Derivative of the Natural Exponential Function**

$$\frac{d}{dx}(e^x) = e^x$$

Thus, the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ at any point is equal to the y-coordinate of the point (see Figure 11).

EXAMPLE 2 Differentiate the function $y = e^{\tan x}$.

SOLUTION To use the Chain Rule, we let $u = \tan x$. Then we have $y = e^u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\frac{du}{dx} = e^{\tan x}\sec^2 x$$

In general if we combine Formula 9 with the Chain Rule, as in Example 2, we get

10 $$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$$

EXAMPLE 3 Find y' if $y = e^{-4x} \sin 5x$.

SOLUTION Using Formula 10 and the Product Rule, we have

$$y' = e^{-4x}(\cos 5x)(5) + (\sin 5x)e^{-4x}(-4) = e^{-4x}(5 \cos 5x - 4 \sin 5x)$$

We have seen that e is a number that lies somewhere between 2 and 3, but we ca use Equation 5 to estimate the numerical value of e more accurately. Let $e = 2^c$. The $e^x = 2^{cx}$. If $f(x) = 2^x$, then from Equation 5 we have $f'(x) = k2^x$, where the value of k i $f'(0) \approx 0.693147$. Thus, by the Chain Rule,

$$e^x = \frac{d}{dx}(e^x) = \frac{d}{dx}(2^{cx}) = k2^{cx}\frac{d}{dx}(cx) = ck2^{cx}$$

Putting $x = 0$, we have $1 = ck$, so $c = 1/k$ and

$$e = 2^{1/k} \approx 2^{1/0.693147} \approx 2.71828$$

It can be shown that the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

The decimal expansion of e is nonrepeating because e is an irrational number.

EXAMPLE 4 In Example 6 in Section 3.4 we considered a population of bacteria cells in a homogeneous nutrient medium. We showed that if the population doubles every hour, then the population after t hours is

$$n = n_0 2^t$$

where n_0 is the initial population. Now we can use (5) and (6) to compute the growth rate:

The rate of growth is proportional to the size of the population.

$$\frac{dn}{dt} \approx n_0(0.693147)2^t$$

For instance, if the initial population is $n_0 = 1000$ cells, then the growth rate after two hours is

$$\left.\frac{dn}{dt}\right|_{t=2} \approx (1000)(0.693147)2^t\big|_{t=2}$$
$$= (4000)(0.693147) \approx 2773 \text{ cells/h}$$

EXAMPLE 5 Find the absolute maximum value of the function $f(x) = xe^{-x}$.

SOLUTION We differentiate to find any critical numbers:

$$f'(x) = xe^{-x}(-1) + e^{-x}(1) = e^{-x}(1 - x)$$

Since exponential functions are always positive, we see that $f'(x) > 0$ when $1 - x > 0$, that is, when $x < 1$. Similarly, $f'(x) < 0$ when $x > 1$. By the First Derivative Test for Absolute Extreme Values, f has an absolute maximum value when $x = 1$ and the value is

$$f(1) = (1)e^{-1} = \frac{1}{e} \approx 0.37$$

Exponential Graphs

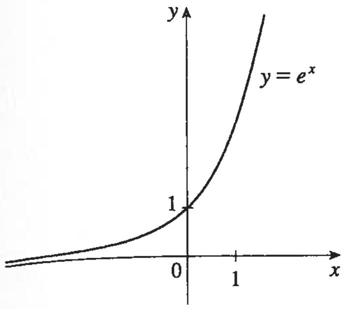


FIGURE 12
The natural exponential function

The exponential function $f(x) = e^x$ is one of the most frequently occurring functions in calculus and its applications, so it is important to be familiar with its graph (Figure 12) and properties. We summarize these properties as follows, using the fact that this function is just a special case of the exponential functions considered in Theorem 2 but with base $a = e > 1$.

> **11 Properties of the Natural Exponential Function** The exponential function $f(x) = e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus, $e^x > 0$ for all x. Also
>
> $$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$$
>
> So the x-axis is a horizontal asymptote of $f(x) = e^x$.

EXAMPLE 6 Find $\lim_{x \to \infty} \dfrac{e^{2x}}{e^{2x} + 1}$.

SOLUTION We divide numerator and denominator by e^{2x}:

$$\lim_{x \to \infty} \frac{e^{2x}}{e^{2x} + 1} = \lim_{x \to \infty} \frac{1}{1 + e^{-2x}} = \frac{1}{1 + \lim_{x \to \infty} e^{-2x}}$$

$$= \frac{1}{1 + 0} = 1$$

We have used the fact that $t = -2x \to -\infty$ as $x \to \infty$ and so

$$\lim_{x \to \infty} e^{-2x} = \lim_{t \to -\infty} e^t = 0$$

EXAMPLE 7 Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.

SOLUTION Notice that the domain of f is $\{x \mid x \neq 0\}$, so we check for vertical asymptotes by computing the left and right limits as $x \to 0$. As $x \to 0^+$, we know that $t = 1/x \to \infty$, so

$$\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty$$

and this shows that $x = 0$ is a vertical asymptote. As $x \to 0^-$, we have $t = 1/x \to -\infty$, so

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

As $x \to \pm\infty$, we have $1/x \to 0$ and so

$$\lim_{x \to \pm\infty} e^{1/x} = e^0 = 1$$

This shows that $y = 1$ is a horizontal asymptote.

Now let's compute the derivative. The Chain Rule gives

$$f'(x) = -\frac{e^{1/x}}{x^2}$$

Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$. Thus, f is decreasing on $(-\infty, 0)$ and on $(0, \infty)$. There is no critical number, so the function has no maximum or minimum. The second derivative is

$$f''(x) = -\frac{x^2e^{1/x}(-1/x^2) - e^{1/x}(2x)}{x^4} = \frac{e^{1/x}(2x+1)}{x^4}$$

Since $e^{1/x} > 0$ and $x^4 > 0$, we have $f''(x) > 0$ when $x > -\frac{1}{2}$ $(x \neq 0)$ and $f''(x) < 0$ when $x < -\frac{1}{2}$. So the curve is concave downward on $\left(-\infty, -\frac{1}{2}\right)$ and concave upward on $\left(-\frac{1}{2}, 0\right)$ and on $(0, \infty)$. The inflection point is $\left(-\frac{1}{2}, e^{-2}\right)$.

To sketch the graph of f we first draw the horizontal asymptote $y = 1$ (as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch [Figure 13(a)]. These parts reflect the information concerning limits and the fact that f is decreasing on both $(-\infty, 0)$ and $(0, \infty)$. Notice that we have indicated that $f(x) \to 0$ as $x \to 0^-$ even though $f(0)$ does not exist. In Figure 13(b) we finish the sketch by incorporating the information concerning concavity and the inflection point. In Figure 13(c) we check our work with a graphing device.

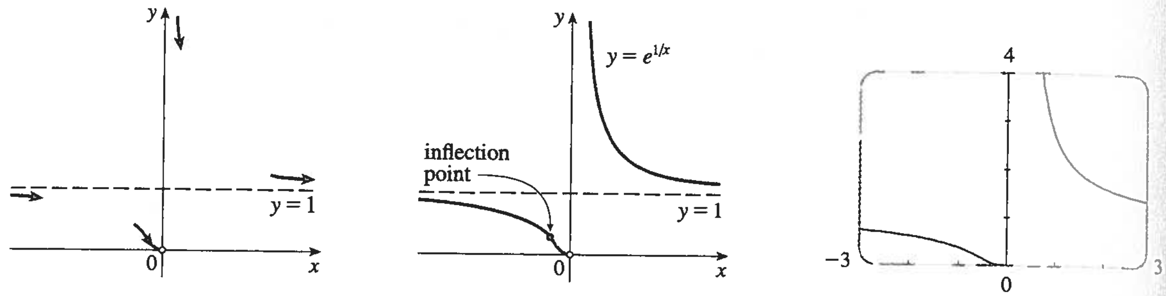


FIGURE 13 (a) Preliminary sketch (b) Finished sketch (c) Computer confirmation

Integration

Because the exponential function $y = e^x$ has a simple derivative, its integral is also simple:

12
$$\int e^x\,dx = e^x + C$$

EXAMPLE 8 Evaluate $\int x^2e^{x^3}\,dx$.

SOLUTION We substitute $u = x^3$. Then $du = 3x^2\,dx$, so $x^2\,dx = \frac{1}{3}\,du$ and

$$\int x^2e^{x^3}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$$

EXAMPLE 9 Find the area under the curve $y = e^{-3x}$ from 0 to 1.

SOLUTION The area is

$$A = \int_0^1 e^{-3x}\,dx = -\frac{1}{3}e^{-3x}\Big]_0^1 = \frac{1}{3}(1 - e^{-3})$$

7.2 Exercises

1. (a) Write an equation that defines the exponential function with base $a > 0$.
   (b) What is the domain of this function?
   (c) If $a \neq 1$, what is the range of this function?
   (d) Sketch the general shape of the graph of the exponential function for each of the following cases.
   (i) $a > 1$ (ii) $a = 1$ (iii) $0 < a < 1$

2. (a) How is the number e defined?
   (b) What is an approximate value for e?
   (c) What is the natural exponential function?

3–6 Graph the given functions on a common screen. How are these graphs related?

3. $y = 2^x, \quad y = e^x, \quad y = 5^x, \quad y = 20^x$

4. $y = e^x, \quad y = e^{-x}, \quad y = 8^x, \quad y = 8^{-x}$

5. $y = 3^x, \quad y = 10^x, \quad y = \left(\frac{1}{3}\right)^x, \quad y = \left(\frac{1}{10}\right)^x$

6. $y = 0.9^x, \quad y = 0.6^x, \quad y = 0.3^x, \quad y = 0.1^x$

7–12 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graphs given in Figures 3 and 12 and, if necessary, the transformations of Section 1.3.

7. $y = 4^x - 3$

8. $y = 4^{x-3}$

9. $y = -2^{-x}$

10. $y = 1 + 2e^x$

11. $y = 3 - e^x$

12. $y = 2 + 5(1 - e^{-x})$

13. Starting with the graph of $y = e^x$, write the equation of the graph that results from
   (a) shifting 2 units downward
   (b) shifting 2 units to the right
   (c) reflecting about the x-axis
   (d) reflecting about the y-axis
   (e) reflecting about the x-axis and then about the y-axis

14. Starting with the graph of $y = e^x$, find the equation of the graph that results from
   (a) reflecting about the line $y = 4$
   (b) reflecting about the line $x = 2$

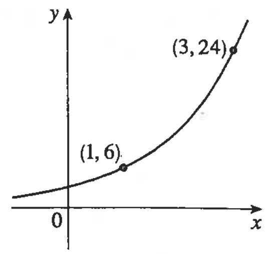
15–16 Find the domain of each function.

15. (a) $f(x) = \dfrac{1}{1 + e^x}$ (b) $f(x) = \dfrac{1}{1 - e^x}$

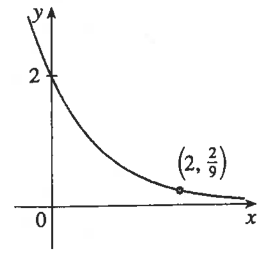
16. (a) $g(t) = \sin(e^{-t})$ (b) $g(t) = \sqrt{1 - 2^t}$

17–18 Find the exponential function $f(x) = Ca^x$ whose graph is given.

17.



18.



19. Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch. Show that, at a distance 2 ft to the right of the origin, the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.

20. Compare the rates of growth of the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place.

21. Compare the functions $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f?

22. Use a graph to estimate the values of x such that $e^x > 1{,}000{,}000{,}000$.

23–28 Find the limit.

23. $\displaystyle\lim_{x\to\infty} (1.001)^x$

24. $\displaystyle\lim_{x\to\infty} e^{-x^2}$

25. $\displaystyle\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

26. $\displaystyle\lim_{x\to(\pi/2)^+} e^{\tan x}$

27. $\displaystyle\lim_{x\to 2^+} e^{3/(2-x)}$

28. $\displaystyle\lim_{x\to 2^-} e^{3/(2-x)}$

29–42 Differentiate the function.

29. $f(x) = x^2 e^x$

30. $y = \dfrac{e^x}{1 + x}$

31. $y = e^{ax^3}$

32. $y = e^u(\cos u + cu)$

33. $f(u) = e^{1/u}$

34. $g(x) = \sqrt{x}\, e^x$

35. $F(t) = e^{t \sin 2t}$

36. $y = e^{k \tan \sqrt{x}}$

37. $y = \sqrt{1 + 2e^{3x}}$

38. $y = \cos(e^{\pi x})$

39. $y = e^{e^x}$

40. $y = \sqrt{1 + xe^{-2x}}$

41. $y = \dfrac{ae^x + b}{ce^x + d}$

42. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

43–44 Find an equation of the tangent line to the curve at the given point.

43. $y = e^{2x} \cos \pi x, \quad (0, 1)$

44. $y = e^x/x, \quad (1, e)$

45. Find y' if $e^{x^2 y} = x + y$.

46. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

47. Show that the function $y = e^x + e^{-x/2}$ satisfies the differential equation $2y'' - y' - y = 0$.

48. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation $y'' + 2y' + y = 0$.

49. For what values of r does the function $y = e^{rx}$ satisfy the equation $y'' + 6y' + 8y = 0$?

50. Find the values of λ for which $y = e^{\lambda x}$ satisfies the equation $y + y' = y''$.

51. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

52. Find the thousandth derivative of $f(x) = xe^{-x}$.

53. (a) Use the Intermediate Value Theorem to show that there is a root of the equation $e^x + x = 0$.
(b) Use Newton's method to find the root of the equation in part (a) correct to six decimal places.

54. Use a graph to find an initial approximation (to one decimal place) to the root of the equation $e^{-x^2} = x^3 + x - 3$. Then use Newton's method to find the root correct to six decimal places.

55. If the initial mass of a sample of ^{90}Sr is 24 mg, then the mass after t years, from Equation 4, is

$$m(t) = 24 \cdot 2^{-t/25}$$

(a) Find the mass remaining after 40 years.
(b) Use Equation 7 and the Chain Rule to estimate the rate at which the mass decays after 40 years.
(c) Use a graphing device to estimate the time required for the mass to be reduced to 5 mg.

56. For the period from 1980 to 2000, the percentage of households in the United States with at least one VCR has been modeled by the function

$$V(t) = \frac{85}{1 + 53e^{-0.5t}}$$

where the time t is measured in years since midyear 1980, so $0 \leq t \leq 20$. Use a graph to estimate the time at which the number of VCRs was increasing most rapidly. Then use derivatives to give a more accurate estimate.

57. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants. [In Section 10.5 we will see that this is a reasonable model for $p(t)$.]
(a) Find $\lim_{t \to \infty} p(t)$.
(b) Find the rate of spread of the rumor.
(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

58. An object is attached to the end of a vibrating spring and its displacement from its equilibrium position is $y = 8e^{-t/2} \sin 4t$, where t is measured in seconds and y is measured in centimeters.
(a) Graph the displacement function together with the functions $y = 8e^{-t/2}$ and $y = -8e^{-t/2}$. How are these graphs related? Can you explain why?
(b) Use the graph to estimate the maximum value of the displacement. Does it occur when the graph touches the graph of $y = 8e^{-t/2}$?
(c) What is the velocity of the object when it first returns to its equilibrium position?
(d) Use the graph to estimate the time after which the displacement is no more than 2 cm from equilibrium.

59. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The following data describe the charge Q remaining on the capacitor (measured in microcoulombs, μC) at time t (measured in seconds).

t	0.00	0.02	0.04	0.06	0.08	0.10
Q	100.00	81.87	67.03	54.88	44.93	36.76

(a) Use a graphing calculator or computer to find an exponential model for the charge. (See Section 1.2.)
(b) The derivative $Q'(t)$ represents the electric current (measured in microamperes, μA) flowing from the capacitor to the flash bulb. Use part (a) to estimate the current when $t = 0.04$ s. Compare with the result of Example 2 in Section 2.1.

60. The table gives the U.S. population from 1790 to 1860.

Year	Population	Year	Population
1790	3,929,000	1830	12,861,000
1800	5,308,000	1840	17,063,000
1810	7,240,000	1850	23,192,000
1820	9,639,000	1860	31,443,000

(a) Use a graphing calculator or computer to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?

(b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.
(c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).
(d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

61. Find the absolute maximum value of the function $f(x) = x - e^x$.

62. Find the absolute minimum value of the function $g(x) = e^x/x$, $x > 0$.

63–64 Find (a) the intervals of increase or decrease, (b) the intervals of concavity, and (c) the points of inflection.

63. $f(x) = xe^x$ **64.** $f(x) = x^2e^x$

65–67 Discuss the curve using the guidelines of Section 4.5.

65. $y = e^{-1/(x+1)}$ **66.** $y = e^{2x} - e^x$

67. $y = e^{3x} + e^{-2x}$

68–69 Draw a graph of f that shows all the important aspects of the curve. Estimate the local maximum and minimum values and then use calculus to find these values exactly. Use a graph of f'' to estimate the inflection points.

68. $f(x) = e^{\cos x}$ **69.** $f(x) = e^{x^3 - x}$

70. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function*. The constant μ is called the *mean* and the positive constant σ is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) Find the asymptote, maximum value, and inflection points of f.
(b) What role does σ play in the shape of the curve?
(c) Illustrate by graphing four members of this family on the same screen.

71–78 Evaluate the integral.

71. $\int_0^5 e^{-3x}\,dx$ **72.** $\int_0^1 xe^{-x^2}\,dx$

73. $\int e^x\sqrt{1 + e^x}\,dx$ **74.** $\int \sec^2 x\, e^{\tan x}\,dx$

75. $\int \frac{e^x + 1}{e^x}\,dx$ **76.** $\int \frac{e^{1/x}}{x^2}\,dx$

77. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ **78.** $\int e^x \sin(e^x)\,dx$

79. Find, correct to three decimal places, the area of the region bounded by the curves $y = e^x$, $y = e^{3x}$, and $x = 1$.

80. Find $f(x)$ if $f''(x) = 3e^x + 5\sin x$, $f(0) = 1$, and $f'(0) = 2$.

81. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

82. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curves $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$.

83. If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$.

84. Evaluate $\lim_{x\to\pi} \frac{e^{\sin x} - 1}{x - \pi}$.

85. (a) Show that $e^x \geqslant 1 + x$ if $x \geqslant 0$.
[*Hint:* Show that $f(x) = e^x - (1 + x)$ is increasing for $x > 0$.]
(b) Deduce that $\frac{4}{3} \leqslant \int_0^1 e^{x^2}\,dx \leqslant e$.

86. (a) Use the inequality of Exercise 85(a) to show that, for $x \geqslant 0$,

$$e^x \geqslant 1 + x + \tfrac{1}{2}x^2$$

(b) Use part (a) to improve the estimate of $\int_0^1 e^{x^2}\,dx$ given in Exercise 85(b).

87. (a) Use mathematical induction to prove that for $x \geqslant 0$ and any positive integer n,

$$e^x \geqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

(b) Use part (a) to show that $e > 2.7$.
(c) Use part (a) to show that

$$\lim_{x\to\infty} \frac{e^x}{x^k} = \infty$$

for any positive integer k.

7.3 Logarithmic Functions

If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing and so it is one-to-one. It therefore has an inverse function f^{-1}, which is called the **logarithmic function with base a** and is denoted by $\log_a$. If we use the formulation of an inverse function given by (7.1.3),

$$f^{-1}(x) = y \iff f(y) = x$$

then we have

1

$$\log_a x = y \iff a^y = x$$

Thus, if $x > 0$, $\log_a x$ is the exponent to which the base a must be raised to give x.

EXAMPLE 1 Evaluate (a) $\log_3 81$, (b) $\log_{25} 5$, and (c) $\log_{10} 0.001$.

SOLUTION

(a) $\log_3 81 = 4$ because $3^4 = 81$

(b) $\log_{25} 5 = \frac{1}{2}$ because $25^{1/2} = 5$

(c) $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$

The cancellation equations (7.1.4), when applied to $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, become

2

$$\log_a(a^x) = x \quad \text{for every } x \in \mathbb{R}$$

$$a^{\log_a x} = x \quad \text{for every } x > 0$$

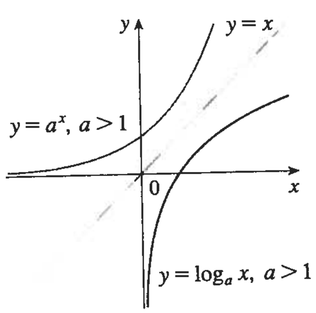


FIGURE 1

The logarithmic function $\log_a$ has domain $(0, \infty)$ and range $\mathbb{R}$ and is continuous since it is the inverse of a continuous function, namely, the exponential function. Its graph is the reflection of the graph of $y = a^x$ about the line $y = x$.

Figure 1 shows the case where $a > 1$. (The most important logarithmic functions have base $a > 1$.) The fact that $y = a^x$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \log_a x$ is a very slowly increasing function for $x > 1$.

Figure 2 shows the graphs of $y = \log_a x$ with various values of the base a. Since $\log_a 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

The following theorem summarizes the properties of logarithmic functions.

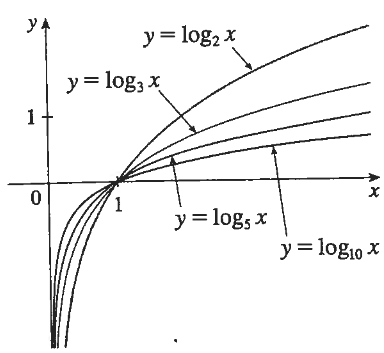


FIGURE 2

3 Theorem If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y > 0$ and r is any real number, then

1. $\log_a(xy) = \log_a x + \log_a y$

2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a(x^r) = r \log_a x$

Properties 1, 2, and 3 follow from the corresponding properties of exponential functions given in Section 7.2.

EXAMPLE 2 Use the properties of logarithms in Theorem 3 to evaluate the following.
(a) $\log_4 2 + \log_4 32$ (b) $\log_2 80 - \log_2 5$

SOLUTION
(a) Using Property 1 in Theorem 3, we have

$$\log_4 2 + \log_4 32 = \log_4(2 \cdot 32) = \log_4 64 = 3$$

since $4^3 = 64$.
(b) Using Property 2 we have

$$\log_2 80 - \log_2 5 = \log_2\left(\tfrac{80}{5}\right) = \log_2 16 = 4$$

since $2^4 = 16$.

The limits of exponential functions given in Section 7.2 are reflected in the following limits of logarithmic functions. (Compare with Figure 1.)

4 If $a > 1$, then

$$\lim_{x\to\infty} \log_a x = \infty \qquad \text{and} \qquad \lim_{x\to 0^+} \log_a x = -\infty$$

In particular, the y-axis is a vertical asymptote of the curve $y = \log_a x$.

EXAMPLE 3 Find $\lim_{x\to 0} \log_{10}(\tan^2 x)$.

SOLUTION As $x \to 0$, we know that $t = \tan^2 x \to \tan^2 0 = 0$ and the values of t are positive. So by (4) with $a = 10 > 1$, we have

$$\lim_{x\to 0} \log_{10}(\tan^2 x) = \lim_{t\to 0^+} \log_{10} t = -\infty$$

Natural Logarithms

||| NOTATION FOR LOGARITHMS
Most textbooks in calculus and the sciences, as well as calculators, use the notation $\ln x$ for the natural logarithm and $\log x$ for the "common logarithm," $\log_{10} x$. In the more advanced mathematical and scientific literature and in computer languages, however, the notation $\log x$ usually denotes the natural logarithm.

Of all possible bases a for logarithms, we will see in the next section that the most convenient choice of a base is the number e, which was defined in Section 7.2. The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

If we put $a = e$ and replace $\log_e$ with "ln" in (1) and (2), then the defining properties of the natural logarithm function become

5
$$\ln x = y \iff e^y = x$$

6
$$\ln(e^x) = x \qquad x \in \mathbb{R}$$
$$e^{\ln x} = x \qquad x > 0$$

In particular, if we set $x = 1$, we get

$$\ln e = 1$$

EXAMPLE 4 Find x if $\ln x = 5$.

SOLUTION 1 From (5) we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore, $x = e^5$.

(If you have trouble working with the "ln" notation, just replace it by $\log_e$. Then the equation becomes $\log_e x = 5$; so, by the definition of logarithm, $e^5 = x$.)

SOLUTION 2 Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in (6) says that $e^{\ln x} = x$. Therefore, $x = e^5$.

EXAMPLE 5 Solve the equation $e^{5-3x} = 10$.

SOLUTION We take natural logarithms of both sides of the equation and use (6):

$$\ln(e^{5-3x}) = \ln 10$$

$$5 - 3x = \ln 10$$

$$3x = 5 - \ln 10$$

$$x = \tfrac{1}{3}(5 - \ln 10)$$

Since the natural logarithm is found on scientific calculators, we can approximate the solution to four decimal places: $x \approx 0.8991$.

EXAMPLE 6 Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

SOLUTION Using Properties 3 and 1 of logarithms, we have

$$\begin{aligned} \ln a + \tfrac{1}{2}\ln b &= \ln a + \ln b^{1/2} \\ &= \ln a + \ln\sqrt{b} \\ &= \ln\left(a\sqrt{b}\right) \end{aligned}$$

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

7 **Change of Base Formula** For any positive number a ($a \neq 1$), we have

$$\log_a x = \frac{\ln x}{\ln a}$$

Proof Let $y = \log_a x$. Then, from (1), we have $a^y = x$. Taking natural logarithms of both sides of this equation, we get $y \ln a = \ln x$. Therefore

$$y = \frac{\ln x}{\ln a}$$

Scientific calculators have a key for natural logarithms, so Formula 7 enables us to use a calculator to compute a logarithm with any base (as shown in the next example). Similarly, Formula 7 allows us to graph any logarithmic function on a graphing calculator or computer (see Exercises 20–22).

EXAMPLE 7 Evaluate $\log_8 5$ correct to six decimal places.

SOLUTION Formula 7 gives

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976$$

EXAMPLE 8 In Section 7.2 we showed that the mass of ^{90}Sr that remains from a 24-mg sample after t years is $m = f(t) = 24 \cdot 2^{-t/25}$. Find the inverse of this function and interpret it.

SOLUTION We need to solve the equation $m = 24 \cdot 2^{-t/25}$ for t. We start by taking natural logarithms of both sides:

$$\ln m = \ln(24 \cdot 2^{-t/25}) = \ln 24 + \ln(2^{-t/25})$$

$$\ln m = \ln 24 - \frac{t}{25} \ln 2$$

$$\frac{t}{25} \ln 2 = \ln 24 - \ln m$$

$$t = \frac{25}{\ln 2}(\ln 24 - \ln m)$$

So the inverse function is

$$f^{-1}(m) = \frac{25}{\ln 2}(\ln 24 - \ln m)$$

This function gives the time required for the mass to decay to m milligrams. In particular, the time required for the mass to be reduced to 5 mg is

$$t = f^{-1}(5) = \frac{25}{\ln 2}(\ln 24 - \ln 5) \approx 56.58 \text{ years}$$

The graphs of the exponential function $y = e^x$ and its inverse function, the natural logarithm function, are shown in Figure 3. Because the curve $y = e^x$ crosses the y-axis with a slope of 1, it follows that the reflected curve $y = \ln x$ crosses the x-axis with a slope of 1.

In common with all other logarithmic functions with base greater than 1, the natural logarithm is a continuous, increasing function defined on $(0, \infty)$ and the y-axis is a vertical asymptote.

If we put $a = e$ in (4), then we have the following limits:

8

$$\lim_{x\to\infty} \ln x = \infty \qquad \lim_{x\to 0^+} \ln x = -\infty$$

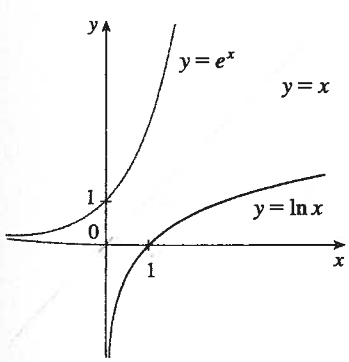


FIGURE 3

EXAMPLE 9 Sketch the graph of the function $y = \ln(x - 2) - 1$.

SOLUTION We start with the graph of $y = \ln x$ as given in Figure 3. Using the transformations of Section 1.3, we shift it 2 units to the right to get the graph of $y = \ln(x - 2)$ and then we shift it 1 unit downward to get the graph of $y = \ln(x - 2) - 1$. (See Figure 4.) Notice that the line $x = 2$ is a vertical asymptote since

$$\lim_{x \to 2^+} [\ln(x - 2) - 1] = -\infty$$

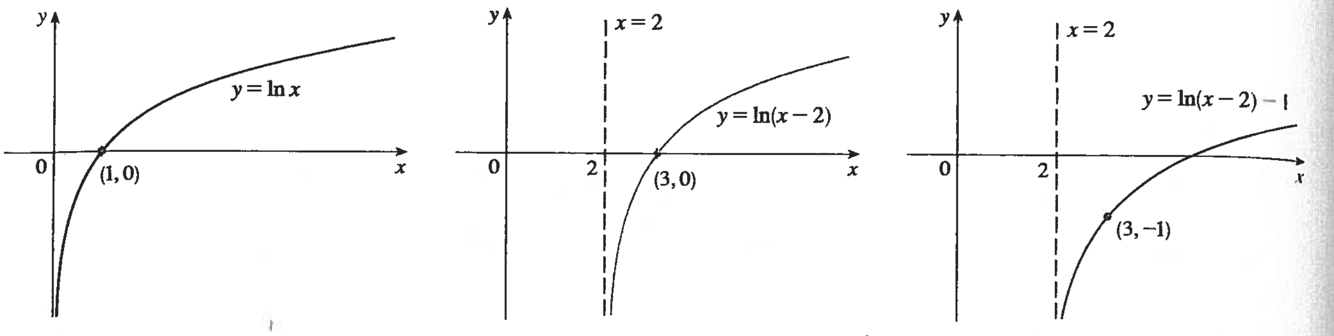


FIGURE 4

We have seen that $\ln x \to \infty$ as $x \to \infty$. But this happens *very* slowly. In fact, $\ln x$ grows more slowly than any positive power of x. To illustrate this fact, we compare approximate values of the functions $y = \ln x$ and $y = x^{1/2} = \sqrt{x}$ in the following table and we graph them in Figures 5 and 6.

x	1	2	5	10	50	100	500	1000	10,000	100,000
$\ln x$	0	0.69	1.61	2.30	3.91	4.6	6.2	6.9	9.2	11.5
$\sqrt{x}$	1	1.41	2.24	3.16	7.07	10.0	22.4	31.6	100	316
$\frac{\ln x}{\sqrt{x}}$	0	0.49	0.72	0.73	0.55	0.46	0.28	0.22	0.09	0.04

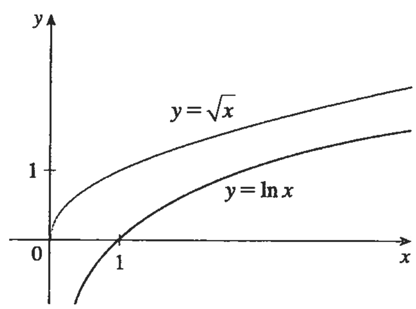


FIGURE 5

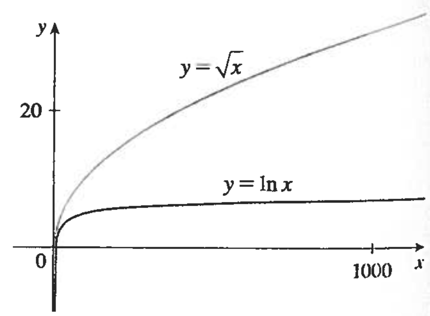


FIGURE 6

You can see that initially the graphs of $y = \sqrt{x}$ and $y = \ln x$ grow at comparable rates, but eventually the root function far surpasses the logarithm. In fact, we will be able to show in Section 7.7 that

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$$

for any positive power p. So for large x, the values of $\ln x$ are very small compared with x^p. (See Exercise 74.)

7.3 Exercises

1. (a) How is the logarithmic function $y = \log_a x$ defined?
(b) What is the domain of this function?
(c) What is the range of this function?
(d) Sketch the general shape of the graph of the function $y = \log_a x$ if $a > 1$.

2. (a) What is the natural logarithm?
(b) What is the common logarithm?
(c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

3–8 Find the exact value of each expression.

3. (a) $\log_{10} 1000$ (b) $\log_{16} 4$
4. (a) $\ln e^{-100}$ (b) $\log_3 81$
5. (a) $\log_5 \frac{1}{25}$ (b) $e^{\ln 15}$
6. (a) $\log_{10} 0.1$ (b) $\log_8 320 - \log_8 5$
7. (a) $\log_{12} 3 + \log_{12} 48$ (b) $\log_2 5 - \log_2 90 + 2\log_2 3$
8. (a) $2^{(\log_2 3 + \log_2 5)}$ (b) $e^{3\ln 2}$

9–12 Use the properties of logarithms to expand the quantity.

9. $\log_2\left(\dfrac{x^3 y}{z^2}\right)$
10. $\ln\sqrt{a(b^2 + c^2)}$
11. $\ln(uv)^{10}$
12. $\ln\dfrac{3x^2}{(x+1)^5}$

13–18 Express the quantity as a single logarithm.

13. $\log_{10} a - \log_{10} b + \log_{10} c$
14. $\ln(x + y) + \ln(x - y) - 2\ln z$
15. $2\ln 4 - \ln 2$
16. $\ln 3 + \frac{1}{3}\ln 8$
17. $\frac{1}{3}\ln x - 5\ln(x^2 + 1)$
18. $\ln x + a\ln y - b\ln z$

19. Use Formula 7 to evaluate each logarithm correct to six decimal places.
(a) $\log_{12} e$ (b) $\log_6 13.54$ (c) $\log_2 \pi$

20–22 Use Formula 7 to graph the given functions on a common screen. How are these graphs related?

20. $y = \log_2 x$, $y = \log_4 x$, $y = \log_6 x$, $y = \log_8 x$
21. $y = \log_{1.5} x$, $y = \ln x$, $y = \log_{10} x$, $y = \log_{50} x$
22. $y = \ln x$, $y = \log_{10} x$, $y = e^x$, $y = 10^x$

23–28 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 1, 2, and 3 and, if necessary, the transformations of Section 1.3.

23. $y = \log_{10}(x + 5)$
24. $y = \log_2(x - 3)$
25. $y = -\ln x$
26. $y = \ln(10x)$
27. $y = 5 + \ln(x - 2)$
28. $y = \ln|x|$

29–38 Solve each equation for x.

29. (a) $2\ln x = 1$ (b) $e^{-x} = 5$
30. (a) $e^{2x+3} - 7 = 0$ (b) $\ln(5 - 2x) = -3$
31. (a) $5^{x-3} = 10$ (b) $\log_{10}(x + 1) = 4$
32. (a) $e^{3x+1} = k$ (b) $\log_2(mx) = c$
33. $\ln(\ln x) = 1$
34. $e^{e^x} = 10$
35. $2\ln x = \ln 2 + \ln(3x - 4)$
36. $\ln(2x + 1) = 2 - \ln x$
37. $e^{ax} = Ce^{bx}$, where $a \neq b$
38. $7e^x - e^{2x} = 12$

39–42 Find the solution of the equation correct to four decimal places.

39. $e^{2+5x} = 100$
40. $\ln(1 + \sqrt{x}) = 2$
41. $\ln(e^x - 2) = 3$
42. $3^{1/(x-4)} = 7$

43–44 Solve each inequality for x.

43. (a) $e^x < 10$ (b) $\ln x > -1$
44. (a) $2 < \ln x < 9$ (b) $e^{2-3x} > 4$

45. Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

46. The velocity of a particle that moves in a straight line under the influence of viscous forces is $v(t) = ce^{-kt}$, where c and k are positive constants.
(a) Show that the acceleration is proportional to the velocity.
(b) Explain the significance of the number c.
(c) At what time is the velocity equal to half the initial velocity?

47. The geologist C. F. Richter defined the magnitude of an earthquake to be $\log_{10}(I/S)$, where I is the intensity of the quake (measured by the amplitude of a seismograph 100 km from the epicenter) and S is the intensity of a "standard" earthquake (where the amplitude is only 1 micron $= 10^{-4}$ cm). The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. The 1906 San Francisco earthquake was 16 times as intense. What was its magnitude on the Richter scale?

48. A sound so faint that it can just be heard has intensity $I_0 = 10^{-12}$ watt/m^2 at a frequency of 1000 hertz (Hz). The loudness, in decibels (dB), of a sound with intensity I is then

defined to be $L = 10 \log_{10}(I/I_0)$. Amplified rock music is measured at 120 dB, whereas the noise from a motor-driven lawn mower is measured at 106 dB. Find the ratio of the intensity of the rock music to that of the mower.

49. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is $n = f(t) = 100 \cdot 2^{t/3}$.
(a) Find the inverse of this function and explain its meaning.
(b) When will the population reach 50,000?

50. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured in seconds.)
(a) Find the inverse of this function and explain its meaning.
(b) How long does it take to recharge the capacitor to 90% of capacity if $a = 2$?

51–56 Find the limit.

51. $\lim_{x \to 2^-} \ln(2 - x)$

52. $\lim_{x \to 3^+} \log_{10}(x^2 - 5x + 6)$

53. $\lim_{x \to 0} \ln(\cos x)$

54. $\lim_{x \to 0^+} \ln(\sin x)$

55. $\lim_{x \to \infty} [\ln(1 + x^2) - \ln(1 + x)]$

56. $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

57–58 Find the domain and range of the function.

57. $f(x) = \log_2(5x - 3)$

58. $G(t) = \ln(e^t - 2)$

59–60 Find (a) the domain of f and (b) f^{-1} and its domain.

59. $f(x) = \sqrt{3 - e^{2x}}$

60. $f(x) = \ln(2 + \ln x)$

61–66 Find the inverse function.

61. $y = \ln(x + 3)$

62. $y = 2^{10^x}$

63. $f(x) = e^{x^3}$

64. $y = (\ln x)^2, \quad x \geq 1$

65. $y = \dfrac{10^x}{10^x + 1}$

66. $y = \dfrac{1 + e^x}{1 - e^x}$

67. On what interval is the function $f(x) = e^{3x} - e^x$ increasing?

68. On what interval is the curve $y = 2e^x - e^{-3x}$ concave downward?

69. (a) Show that the function $f(x) = \ln(x + \sqrt{x^2 + 1})$ is an odd function.
(b) Find the inverse function of f.

70. Find an equation of the tangent to the curve $y = e^{-x}$ that is perpendicular to the line $2x - y = 8$.

71. Show that the equation $x^{1/\ln x} = 2$ has no solution. What can you say about the function $f(x) = x^{1/\ln x}$?

72. Any function of the form $f(x) = [g(x)]^{h(x)}$, where $g(x) > 0$, can be analyzed as a power of e by writing $g(x) = e^{\ln g(x)}$ so that $f(x) = e^{h(x) \ln g(x)}$. Using this device, calculate each limit.
(a) $\lim_{x \to \infty} x^{\ln x}$
(b) $\lim_{x \to 0^+} x^{-\ln x}$
(c) $\lim_{x \to 0^+} x^{1/x}$
(d) $\lim_{x \to \infty} (\ln 2x)^{-\ln x}$

73. Let $a > 1$. Prove, using Definitions 4.4.6 and 4.4.7, that
(a) $\lim_{x \to -\infty} a^x = 0$
(b) $\lim_{x \to \infty} a^x = \infty$

74. (a) Compare the rates of growth of $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?
(b) Graph the function $h(x) = (\ln x)/x^{0.1}$ in a viewing rectangle that displays the behavior of the function as $x \to \infty$.
(c) Find a number N such that

$$\frac{\ln x}{x^{0.1}} < 0.1 \qquad \text{whenever} \qquad x > N$$

75. Solve the inequality $\ln(x^2 - 2x - 2) \leq 0$.

76. A **prime number** is a positive integer that has no factors other than 1 and itself. The first few primes are 2, 3, 5, 7, 11, 13, 17, ... We denote by $\pi(n)$ the number of primes that are less than or equal to n. For instance, $\pi(15) = 6$ because there are six primes smaller than 15.
(a) Calculate the numbers $\pi(25)$ and $\pi(100)$.
[*Hint:* To find $\pi(100)$, first compile a list of the primes up to 100 using the *sieve of Eratosthenes:* Write the numbers from 2 to 100 and cross out all multiples of 2. Then cross out all multiples of 3. The next remaining number is 5, so cross out all remaining multiples of it, and so on.]
(b) By inspecting tables of prime numbers and tables of logarithms, the great mathematician K. F. Gauss made the guess in 1792 (when he was 15) that the number of primes up to n is approximately $n/\ln n$ when n is large. More precisely, he conjectured that

$$\lim_{n \to \infty} \frac{\pi(n)}{n/\ln n} = 1$$

This was finally proved, a hundred years later, by Jacques Hadamard and Charles de la Vallée Poussin and is called the **Prime Number Theorem**. Provide evidence for the truth of this theorem by computing the ratio of $\pi(n)$ to $n/\ln n$ for $n = 100, 1000, 10^4, 10^5, 10^6$, and 10^7. Use the following data: $\pi(1000) = 168$, $\pi(10^4) = 1229$, $\pi(10^5) = 9592$, $\pi(10^6) = 78{,}498$, $\pi(10^7) = 664{,}579$.
(c) Use the Prime Number Theorem to estimate the number of primes up to a billion.

7.4 Derivatives of Logarithmic Functions

In this section we find the derivatives of the logarithmic functions $y = \log_a x$ and the exponential functions $y = a^x$. We start with the natural logarithmic function $y = \ln x$. We know that it is differentiable because it is the inverse of the differentiable function $y = e^x$.

1

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Proof Let $y = \ln x$. Then

$$e^y = x$$

Differentiating this equation implicitly with respect to x, we get

$$e^y \frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$

In general, if we combine Formula 1 with the Chain Rule as in Example 1, we get

2

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

EXAMPLE 2 Find $\dfrac{d}{dx}\ln(\sin x)$.

SOLUTION Using (2), we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$

EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

EXAMPLE 4 Find $\dfrac{d}{dx}\ln\dfrac{x+1}{\sqrt{x-2}}$.

SOLUTION 1

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{1}{\dfrac{x+1}{\sqrt{x-2}}}\frac{d}{dx}\frac{x+1}{\sqrt{x-2}}$$

$$= \frac{\sqrt{x-2}}{x+1}\,\frac{\sqrt{x-2}\cdot 1 - (x+1)(\frac{1}{2})(x-2)^{-1/2}}{x-2}$$

$$= \frac{x-2-\frac{1}{2}(x+1)}{(x+1)(x-2)} = \frac{x-5}{2(x+1)(x-2)}$$

Figure 1 shows the graph of the function f of Example 4 together with the graph of its derivative. It gives a visual check on our calculation. Notice that $f'(x)$ is large negative when f is rapidly decreasing and $f'(x) = 0$ when f has a minimum.

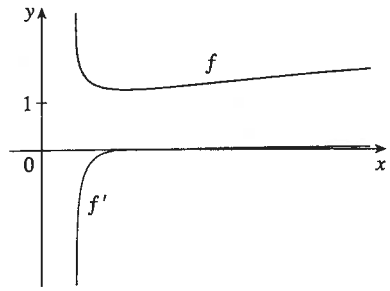


FIGURE 1

SOLUTION 2 If we first simplify the given function using the laws of logarithms, then the differentiation becomes easier:

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left[\ln(x+1) - \tfrac{1}{2}\ln(x-2)\right]$$

$$= \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.)

EXAMPLE 5 Find the absolute minimum value of $f(x) = x^2\ln x$.

SOLUTION The domain is $(0, \infty)$ and the Product Rule gives

$$f'(x) = x^2\cdot\frac{1}{x} + 2x\ln x = x(1 + 2\ln x)$$

Therefore, $f'(x) = 0$ when $2\ln x = -1$, that is, $\ln x = -\frac{1}{2}$, or $x = e^{-1/2}$. Also, $f'(x) > 0$ when $x > e^{-1/2}$ and $f'(x) < 0$ for $0 < x < e^{-1/2}$. So by the First Derivative Test for Absolute Extreme Values, $f(1/\sqrt{e}) = -1/(2e)$ is the absolute minimum.

EXAMPLE 6 Discuss the curve $y = \ln(4 - x^2)$ using the guidelines of Section 4.5.

SOLUTION

A. The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = \log_e 1 = 0$ (since $e^0 = 1$), so we have $4 - x^2 = 1 \Rightarrow x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

C. Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

D. We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x\to 2^-}\ln(4 - x^2) = -\infty \qquad \text{and} \qquad \lim_{x\to -2^+}\ln(4 - x^2) = -\infty$$

Thus, the lines $x = 2$ and $x = -2$ are vertical asymptotes.

E.
$$f'(x) = \frac{-2x}{4 - x^2}$$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G.
$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.

H. Using this information, we sketch the curve in Figure 2.

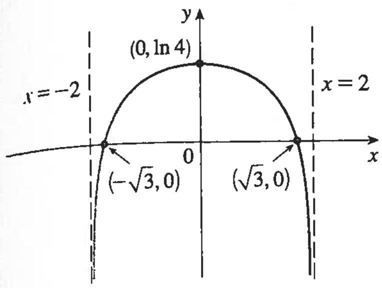


FIGURE 2
$y = \ln(4 - x^2)$

EXAMPLE 7 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus, $f'(x) = 1/x$ for all $x \neq 0$.

The result of Example 7 is worth remembering:

3
$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

The corresponding integration formula is

4
$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Notice that this fills the gap in the rule for integrating power functions:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \text{if } n \neq -1$$

The missing case $(n = -1)$ is supplied by Formula 4.

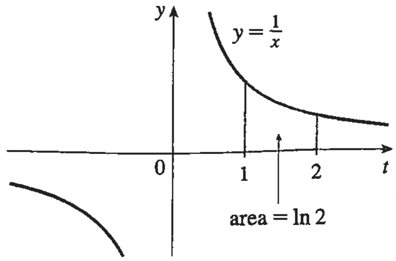


FIGURE 3

EXAMPLE 8 Find, correct to three decimal places, the area of the region under the hyperbola $xy = 1$ from $x = 1$ to $x = 2$.

SOLUTION The given region is shown in Figure 3. Using Formula 4 (without the absolute value sign, since $x > 0$), we see that the area is

$$A = \int_1^2 \frac{1}{x}\,dx = \ln x\Big]_1^2$$
$$= \ln 2 - \ln 1 = \ln 2 \approx 0.693$$

EXAMPLE 9 Evaluate $\displaystyle\int \frac{x}{x^2 + 1}\,dx$.

SOLUTION We make the substitution $u = x^2 + 1$ because the differential $du = 2x\,dx$ occurs (except for the constant factor 2). Thus, $x\,dx = \frac{1}{2}\,du$ and

$$\int \frac{x}{x^2 + 1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}\ln|x^2 + 1| + C = \frac{1}{2}\ln(x^2 + 1) + C$$

Notice that we removed the absolute value signs because $x^2 + 1 > 0$ for all x. We could use the properties of logarithms to write the answer as

$$\ln\sqrt{x^2 + 1} + C$$

but this isn't necessary.

IIII Since the function $f(x) = (\ln x)/x$ in Example 10 is positive for $x > 1$, the integral represents the area of the shaded region in Figure 4.

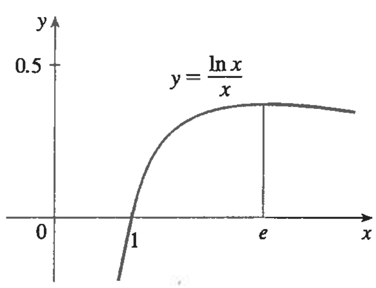


FIGURE 4

EXAMPLE 10 Calculate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.

SOLUTION We let $u = \ln x$ because its differential $du = dx/x$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}$$

EXAMPLE 11 Calculate $\displaystyle\int \tan x\,dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

This suggests that we should substitute $u = \cos x$ since then $du = -\sin x\,dx$ and so $\sin x\,dx = -du$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u}$$
$$= -\ln|u| + C = -\ln|\cos x| + C$$

Since $-\ln|\cos x| = \ln(1/|\cos x|) = \ln|\sec x|$, the result of Example 11 can also be written as

5 $$\int \tan x\, dx = \ln|\sec x| + C$$

General Logarithmic and Exponential Functions

Formula 7 in Section 7.3 expresses a logarithmic function with base a in terms of the natural logarithmic function:

$$\log_a x = \frac{\ln x}{\ln a}$$

Since $\ln a$ is a constant, we can differentiate as follows:

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{\ln a}\frac{d}{dx}(\ln x) = \frac{1}{x \ln a}$$

6 $$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

EXAMPLE 12

$$\frac{d}{dx}\log_{10}(2 + \sin x) = \frac{1}{(2 + \sin x)\ln 10}\frac{d}{dx}(2 + \sin x) = \frac{\cos x}{(2 + \sin x)\ln 10}$$

From Formula 6 we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: The differentiation formula is simplest when $a = e$ because $\ln e = 1$.

Exponential Functions with Base a In Section 7.2 we showed that the derivative of the general exponential function $f(x) = a^x$, $a > 0$, is a constant multiple of itself:

$$f'(x) = f'(0)a^x \qquad \text{where} \qquad f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}$$

We are now in a position to show that the value of the constant is $f'(0) = \ln a$.

7 $$\frac{d}{dx}(a^x) = a^x \ln a$$

Proof We use the fact that $e^{\ln a} = a$:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{\ln a})^x = \frac{d}{dx}e^{(\ln a)x} = e^{(\ln a)x}\frac{d}{dx}(\ln a)x$$
$$= (e^{\ln a})^x(\ln a) = a^x \ln a$$

In Example 6 in Section 3.4 we considered a population of bacteria cells that doubles every hour and saw that the population after t hours is $n = n_0 2^t$, where n_0 is the initial population. Formula 7 enables us to find the growth rate:

$$\frac{dn}{dt} = n_0 2^t \ln 2$$

EXAMPLE 13 Combining Formula 7 with the Chain Rule, we have

$$\frac{d}{dx}(10^{x^2}) = 10^{x^2}(\ln 10)\frac{d}{dx}(x^2) = (2 \ln 10)x10^{x^2}$$

The integration formula that follows from Formula 7 is

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad a \neq 1$$

EXAMPLE 14
$$\int_0^5 2^x\,dx = \frac{2^x}{\ln 2}\bigg]_0^5 = \frac{2^5}{\ln 2} - \frac{2^0}{\ln 2} = \frac{31}{\ln 2}$$

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE 15 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use properties of logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

If we hadn't used logarithmic differentiation in Example 15, we would have had to use both the Quotient Rule and the Product Rule. The resulting calculation would have been horrendous.

Steps in Logarithmic Differentiation

1. Take logarithms of both sides of an equation $y = f(x)$ and use properties of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined, but we can write $|y| = |f(x)|$ and use Equation 3. We illustrate this procedure by proving the general version of the Power Rule, as promised in Section 3.3.

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of a derivative.

Proof Let $y = x^n$ and use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n\ln|x| \qquad x \neq 0$$

Therefore

$$\frac{y'}{y} = \frac{n}{x}$$

Hence

$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

You should distinguish carefully between the Power Rule $[(d/dx)\,x^n = nx^{n-1}]$, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions $[(d/dx)\,a^x = a^x \ln a]$, where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

1. $\dfrac{d}{dx}(a^b) = 0 \qquad$ (a and b are constants)
2. $\dfrac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$
3. $\dfrac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x)$
4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

Figure 5 illustrates Example 16 by showing the graphs of $f(x) = x^{\sqrt{x}}$ and its derivative.

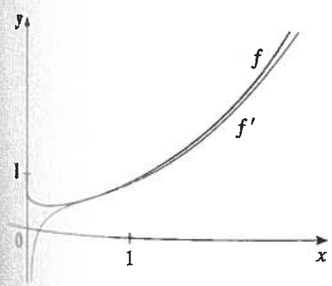


FIGURE 5

EXAMPLE 16 Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1 Using logarithmic differentiation, we have

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$$

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

SOLUTION 2 Another method is to write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$:

$$\frac{d}{dx}\left(x^{\sqrt{x}}\right) = \frac{d}{dx}\left(e^{\sqrt{x}\ln x}\right) = e^{\sqrt{x}\ln x}\frac{d}{dx}\left(\sqrt{x}\ln x\right)$$

$$= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right) \qquad \text{(as in Solution 1)}$$

The Number e as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus, $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{x\to 0}\frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x\to 0}\frac{\ln(1+x) - \ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x)$$

$$= \lim_{x\to 0}\ln(1+x)^{1/x}$$

Because $f'(1) = 1$, we have

$$\lim_{x\to 0}\ln(1+x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential functions, we have

$$e = e^1 = e^{\lim_{x\to 0}\ln(1+x)^{1/x}} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = \lim_{x\to 0}(1+x)^{1/x}$$

8
$$e = \lim_{x\to 0}(1+x)^{1/x}$$

Formula 8 is illustrated by the graph of the function $y = (1+x)^{1/x}$ in Figure 6 and a table of values for small values of x.

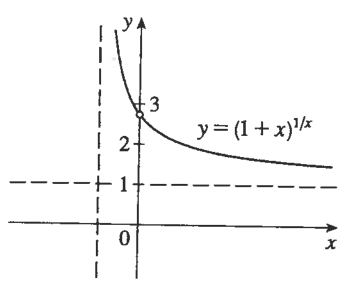


FIGURE 6

x	$(1+x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

If we put $n = 1/x$ in Formula 8, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

9
$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

7.4 Exercises

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_a x$.

2–24 Differentiate the function.

2. $f(x) = \ln(x^2 + 10)$

3. $f(\theta) = \ln(\cos\theta)$

4. $f(x) = \cos(\ln x)$

5. $f(x) = \log_2(1 - 3x)$

6. $f(x) = \log_{10}\left(\dfrac{x}{x-1}\right)$

7. $f(x) = \sqrt[5]{\ln x}$

8. $f(x) = \ln\sqrt[5]{x}$

9. $f(x) = \sqrt{x}\ln x$

10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$

11. $F(t) = \ln\dfrac{(2t+1)^3}{(3t-1)^4}$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

13. $g(x) = \ln\dfrac{a - x}{a + x}$

14. $F(y) = y\ln(1 + e^y)$

15. $f(u) = \dfrac{\ln u}{1 + \ln(2u)}$

16. $y = \ln(x^4\sin^2 x)$

17. $h(t) = t^3 - 3^t$

18. $y = 10^{\tan\theta}$

19. $y = \ln|2 - x - 5x^2|$

20. $G(u) = \ln\sqrt{\dfrac{3u+2}{3u-2}}$

21. $y = \ln(e^{-x} + xe^{-x})$

22. $y = [\ln(1 + e^x)]^2$

23. $y = 5^{-1/x}$

24. $y = 2^{3^{x^2}}$

25–28 Find y' and y''.

25. $y = x\ln x$

26. $y = \dfrac{\ln x}{x^2}$

27. $y = \log_{10} x$

28. $y = \ln(\sec x + \tan x)$

29–32 Differentiate f and find the domain of f.

29. $f(x) = \dfrac{x}{1 - \ln(x-1)}$

30. $f(x) = \dfrac{1}{1 + \ln x}$

31. $f(x) = x^2\ln(1 - x^2)$

32. $f(x) = \ln\ln\ln x$

33. If $f(x) = \dfrac{x}{\ln x}$, find $f'(e)$.

34. If $f(x) = x^2\ln x$, find $f'(1)$.

35–36 Find an equation of the tangent line to the curve at the given point.

35. $y = \ln\ln x$, $(e, 0)$

36. $y = \ln(x^3 - 7)$, $(2, 0)$

37–38 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

37. $f(x) = \sin x + \ln x$

38. $f(x) = x^{\cos x}$

39–50 Use logarithmic differentiation to find the derivative of the function.

39. $y = (2x+1)^5(x^4 - 3)^6$

40. $y = \sqrt{x}\,e^{x^2}(x^2+1)^{10}$

41. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2+1)^2}$

42. $y = \sqrt[4]{\dfrac{x^2+1}{x^2-1}}$

43. $y = x^x$

44. $y = x^{1/x}$

45. $y = x^{\sin x}$

46. $y = (\sin x)^x$

47. $y = (\ln x)^x$

48. $y = x^{\ln x}$

49. $y = x^{e^x}$

50. $y = (\ln x)^{\cos x}$

51. Find y' if $y = \ln(x^2 + y^2)$.

52. Find y' if $x^y = y^x$.

53. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x-1)$.

54. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

55–56 Use a graph to estimate the roots of the equation. Then use these estimates as the initial approximations in Newton's method to find the roots correct to six decimal places.

55. $(x-4)^2 = \ln x$

56. $\ln(4 - x^2) = x$

57. Find the intervals of concavity and the inflection points of the function $f(x) = (\ln x)/\sqrt{x}$.

58. Find the absolute minimum value of the function $f(x) = x \ln x$.

59–62 Discuss the curve under the guidelines of Section 4.5.

59. $y = \ln(\sin x)$

60. $y = \ln(\tan^2 x)$

61. $y = \ln(1 + x^2)$

62. $y = \ln(x^2 - 3x + 2)$

CAS 63. If $f(x) = \ln(2x + x \sin x)$, use the graphs of f, f', and f'' to estimate the intervals of increase and the inflection points of f on the interval $(0, 15]$.

64. Investigate the family of curves $f(x) = \ln(x^2 + c)$. What happens to the inflection points and asymptotes as c changes? Graph several members of the family to illustrate what you discover.

65–76 Evaluate the integral.

65. $\displaystyle\int_2^4 \frac{3}{x}\,dx$

66. $\displaystyle\int_1^2 \frac{4+u^2}{u^3}\,du$

67. $\displaystyle\int_1^2 \frac{dt}{8-3t}$

68. $\displaystyle\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

69. $\displaystyle\int_1^e \frac{x^2+x+1}{x}\,dx$

70. $\displaystyle\int_e^6 \frac{dx}{x \ln x}$

71. $\displaystyle\int \frac{2-x^2}{6x-x^3}\,dx$

72. $\displaystyle\int \frac{\cos x}{2+\sin x}\,dx$

73. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

74. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

75. $\displaystyle\int_1^2 10^t\,dt$

76. $\displaystyle\int x 2^{x^2}\,dx$

77. Show that $\int \cot x\,dx = \ln|\sin x| + C$ by (a) differentiating the right side of the equation and (b) using the method of Example 11.

78. Find, correct to three decimal places, the area of the region above the hyperbola $y = 2/(x-2)$, below the x-axis, and between the lines $x = -4$ and $x = -1$.

79. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{\sqrt{x+1}}$$

from 0 to 1 about the x-axis.

80. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{x^2+1}$$

from 0 to 3 about the y-axis.

81. The work done by a gas when it expands from volume V_1 to volume V_2 is $W = \int_{V_1}^{V_2} P\,dV$, where $P = P(V)$ is the pressure as a function of the volume V. (See Exercise 27 in Section 6.4.) Boyle's Law states that when a quantity of gas expands at constant temperature, $PV = C$, where C is a constant. If the initial volume is 600 cm^3 and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm^3.

82. Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$.

83. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

84. If $f(x) = e^x + \ln x$ and $h(x) = f^{-1}(x)$, find $h'(e)$.

85. For what values of m do the line $y = mx$ and the curve $y = x/(x^2+1)$ enclose a region? Find the area of the region.

86. (a) Find the linear approximation to $f(x) = \ln x$ near 1.
(b) Illustrate part (a) by graphing f and its linearization.
(c) For what values of x is the linear approximation accurate to within 0.1?

87. Use the definition of derivative to prove that

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

88. Show that $\displaystyle\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

7.5 Inverse Trigonometric Functions

In this section we apply the ideas of Section 7.1 to find the derivatives of the so-called inverse trigonometric functions. We have a slight difficulty in this task: Because the trigonometric functions are not one-to-one, they do not have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one.

You can see from Figure 1 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test). But the function $f(x) = \sin x$, $-\pi/2 \leqslant x \leqslant \pi/2$ (see Figure 2), *is* one-to-one. The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.

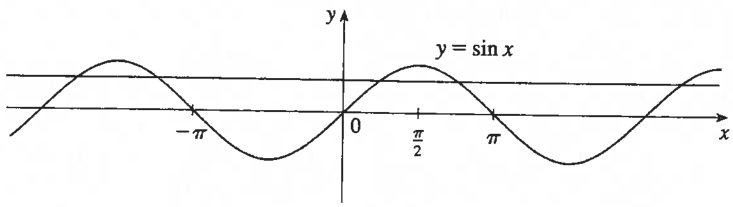


FIGURE 1

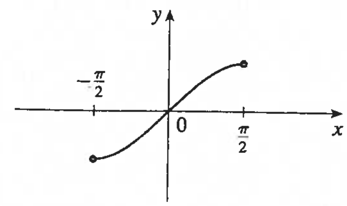


FIGURE 2 $y = \sin x, -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Since the definition of an inverse function says that

$$f^{-1}(x) = y \iff f(y) = x$$

we have

1
$$\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

$\sin^{-1}x \neq \dfrac{1}{\sin x}$

Thus, if $-1 \leqslant x \leqslant 1$, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is x.

EXAMPLE 1 Evaluate (a) $\sin^{-1}(\frac{1}{2})$ and (b) $\tan(\arcsin \frac{1}{3})$.

SOLUTION
(a) We have

$$\sin^{-1}(\tfrac{1}{2}) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

(b) Let $\theta = \arcsin \frac{1}{3}$, so $\sin \theta = \frac{1}{3}$. Then we can draw a right triangle with angle θ as in Figure 3 and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9 - 1} = 2\sqrt{2}$. This enables us to read from the triangle that

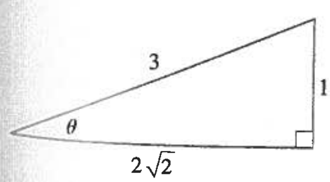


FIGURE 3

$$\tan(\arcsin \tfrac{1}{3}) = \tan \theta = \frac{1}{2\sqrt{2}}$$

The cancellation equations for inverse functions become, in this case,

2

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

We must be careful when using the first cancellation equation because it is valid onl when x lies in the interval $[-\pi/2, \pi/2]$. The following example shows how to procee when x lies outside this interval.

EXAMPLE 2 Evaluate:

(a) $\sin(\sin^{-1} 0.6)$ (b) $\sin^{-1}\left(\sin \frac{\pi}{12}\right)$ (c) $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$

SOLUTION

(a) Since 0.6 lies between -1 and 1, the second cancellation equation in (2) gives

$$\sin(\sin^{-1} 0.6) = 0.6$$

(b) Since $\pi/12$ lies between $-\pi/2$ and $\pi/2$, the first cancellation equation gives

$$\sin^{-1}\left(\sin \frac{\pi}{12}\right) = \frac{\pi}{12}$$

(c) Since $2\pi/3$ does not lie in the interval $[-\pi/2, \pi/2]$, we can't use the cancellation equation. Instead we note that $\sin(2\pi/3) = \sqrt{3}/2$ and $\sin^{-1}(\sqrt{3}/2) = \pi/3$ because $\pi/3$ lies between $-\pi/2$ and $\pi/2$. Therefore

$$\sin^{-1}\left(\sin \frac{2\pi}{3}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

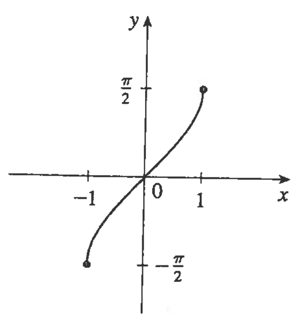


FIGURE 4 $y = \sin^{-1}x = \arcsin x$

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$, and its graph, shown in Figure 4, is obtained from that of the restricted sine function (Figure 2) by reflection about the line $y = x$.

We know that the sine function f is continuous, so the inverse sine function is also continuous. We also know from Section 3.5 that the sine function is differentiable, so the inverse sine function is also differentiable. We could calculate the derivative of $\sin^{-1}$ by the formula in Theorem 7.1.7, but since we know that $\sin^{-1}$ is differentiable, we can just as easily calculate it by implicit differentiation as follows.

Let $y = \sin^{-1}x$. Then $\sin y = x$ and $-\pi/2 \leqslant y \leqslant \pi/2$. Differentiating $\sin y = x$ implicitly with respect to x, we obtain

$$\cos y \frac{dy}{dx} = 1$$

and

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \geqslant 0$ since $-\pi/2 \leqslant y \leqslant \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Therefore $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

3 $$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

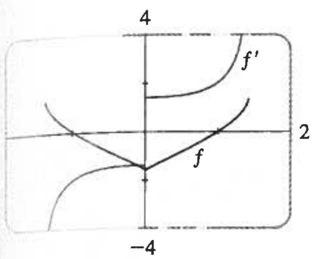


FIGURE 5

||| The graphs of the function f of Example 3 and its derivative are shown in Figure 5. Notice that f is not differentiable at 0 and this is consistent with the fact that the graph of f' makes a sudden jump at $x = 0$.

EXAMPLE 3 If $f(x) = \sin^{-1}(x^2 - 1)$, find (a) the domain of f, (b) $f'(x)$, and (c) the domain of f'.

SOLUTION

(a) Since the domain of the inverse sine function is $[-1, 1]$, the domain of f is

$$\{x \mid -1 \leqslant x^2 - 1 \leqslant 1\} = \{x \mid 0 \leqslant x^2 \leqslant 2\}$$
$$= \{x \mid |x| \leqslant \sqrt{2}\} = [-\sqrt{2}, \sqrt{2}]$$

(b) Combining Formula 3 with the Chain Rule, we have

$$f'(x) = \frac{1}{\sqrt{1 - (x^2 - 1)^2}} \frac{d}{dx}(x^2 - 1)$$
$$= \frac{1}{\sqrt{1 - (x^4 - 2x^2 + 1)}} 2x = \frac{2x}{\sqrt{2x^2 - x^4}}$$

(c) The domain of f' is

$$\{x \mid -1 < x^2 - 1 < 1\} = \{x \mid 0 < x^2 < 2\}$$
$$= \{x \mid 0 < |x| < \sqrt{2}\} = (-\sqrt{2}, 0) \cup (0, \sqrt{2})$$

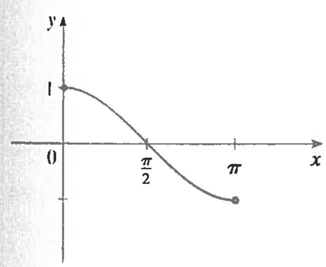


FIGURE 6
$y = \cos x,\ 0 \leqslant x \leqslant \pi$

The **inverse cosine function** is handled similarly. The restricted cosine function $f(x) = \cos x$, $0 \leqslant x \leqslant \pi$, is one-to-one (see Figure 6) and so it has an inverse function denoted by $\cos^{-1}$ or arccos.

4 $$\cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi$$

The cancellation equations are

5 $$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leqslant x \leqslant \pi$$
$$\cos(\cos^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

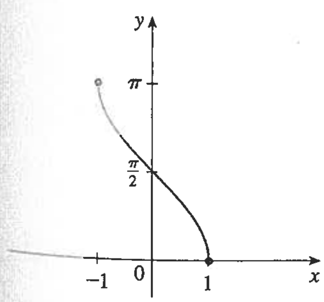


FIGURE 7
$y = \cos^{-1} x = \arccos x$

The inverse cosine function, $\cos^{-1}$, has domain $[-1, 1]$ and range $[0, \pi]$ and is a continuous function whose graph is shown in Figure 7. Its derivative is given by

6 $$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

Formula 6 can be proved by the same method as for Formula 3 and is left as Exercise 17.

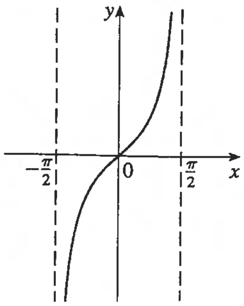


FIGURE 8
$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2, \pi/2)$. Thus, the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$. (See Figure 8.) It is denoted by $\tan^{-1}$ or arctan.

7
$$\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

EXAMPLE 4 Simplify the expression $\cos(\tan^{-1}x)$.

SOLUTION 1 Let $y = \tan^{-1}x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\sec y = \sqrt{1 + x^2} \qquad (\text{since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2)$$

Thus
$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

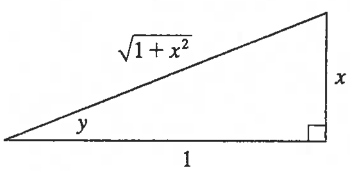


FIGURE 9

SOLUTION 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1}x$, then $\tan y = x$, and we can read from Figure 9 (which illustrates the case $y > 0$) that

$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$

The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure 10.

We know that

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \quad \text{and} \quad \lim_{x \to -(\pi/2)^+} \tan x = -\infty$$

and so the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan. Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$. This fact is expressed by the following limits:

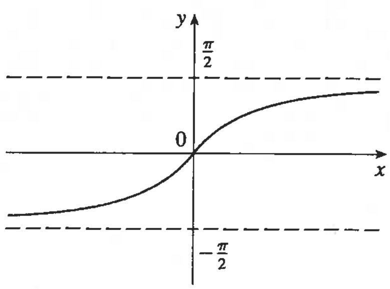


FIGURE 10
$y = \tan^{-1}x = \arctan x$

8
$$\lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2} \qquad \lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2}$$

EXAMPLE 5 Evaluate $\displaystyle\lim_{x \to 2^+} \arctan\left(\frac{1}{x - 2}\right)$.

SOLUTION Since

$$\frac{1}{x - 2} \to \infty \quad \text{as } x \to 2^+$$

the first equation in (8) gives

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x - 2}\right) = \frac{\pi}{2}$$

Since tan is differentiable, $\tan^{-1}$ is also differentiable. To find its derivative, let $y = \tan^{-1}x$. Then $\tan y = x$. Differentiating this latter equation implicitly with respect to x, we have

$$\sec^2 y \frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

9

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$$

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

10 $y = \csc^{-1}x \ (|x| \geqslant 1) \iff \csc y = x$ and $y \in (0, \pi/2] \cup (\pi, 3\pi/2]$

$y = \sec^{-1}x \ (|x| \geqslant 1) \iff \sec y = x$ and $y \in [0, \pi/2) \cup [\pi, 3\pi/2)$

$y = \cot^{-1}x \ (x \in \mathbb{R}) \iff \cot y = x$ and $y \in (0, \pi)$

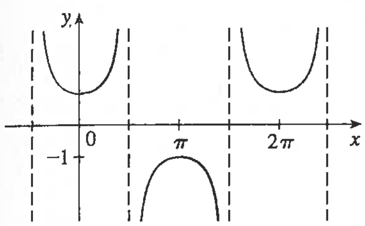


FIGURE 11

$y = \sec x$

The choice of intervals for y in the definitions of $\csc^{-1}$ and $\sec^{-1}$ is not universally agreed upon. For instance, some authors use $y \in [0, \pi/2) \cup (\pi/2, \pi]$ in the definition of $\sec^{-1}$. [You can see from the graph of the secant function in Figure 11 that both this choice and the one in (10) will work.] The reason for the choice in (10) is that the differentiation formulas are simpler (see Exercise 79).

We collect in Table 11 the differentiation formulas for all of the inverse trigonometric functions. The proofs of the formulas for the derivatives of $\csc^{-1}$, $\sec^{-1}$, and $\cot^{-1}$ are left as Exercises 19–21.

11 **Table of Derivatives of Inverse Trigonometric Functions**

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$$

Each of these formulas can be combined with the Chain Rule. For instance, if u is a differentiable function of x, then

$$\frac{d}{dx}(\sin^{-1}u) = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx} \qquad \text{and} \qquad \frac{d}{dx}(\tan^{-1}u) = \frac{1}{1 + u^2}\frac{du}{dx}$$

EXAMPLE 6 Differentiate (a) $y = \dfrac{1}{\sin^{-1}x}$ and (b) $f(x) = x\tan^{-1}\sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$
$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1-x^2}}$$

(b)
$$f'(x) = \tan^{-1}\sqrt{x} + x\frac{1}{1+(\sqrt{x})^2}\,\frac{1}{2}x^{-1/2}$$
$$= \tan^{-1}\sqrt{x} + \frac{\sqrt{x}}{2(1+x)}$$

EXAMPLE 7 Prove the identity $\tan^{-1}x + \cot^{-1}x = \pi/2$.

SOLUTION Although calculus is not needed to prove this identity, the proof using calculus is quite simple. If $f(x) = \tan^{-1}x + \cot^{-1}x$, then

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

for all values of x. Therefore, $f(x) = C$, a constant. To determine the value of C, we put $x = 1$. Then

$$C = f(1) = \tan^{-1}1 + \cot^{-1}1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Thus, $\tan^{-1}x + \cot^{-1}x = \pi/2$.

Each of the formulas in Table 11 gives rise to an integration formula. The two most useful of these are the following:

12
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

13
$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$$

EXAMPLE 8 Find $\displaystyle\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}}\,dx$.

SOLUTION If we write

$$\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}}\,dx = \int_0^{1/4} \frac{1}{\sqrt{1-(2x)^2}}\,dx$$

then the integral resembles Equation 12 and the substitution $u = 2x$ is suggested. This gives $du = 2\,dx$, so $dx = du/2$. When $x = 0$, $u = 0$; when $x = \frac{1}{4}$, $u = \frac{1}{2}$. So

$$\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}}\,dx = \tfrac{1}{2}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}} = \tfrac{1}{2}\sin^{-1}u\Big]_0^{1/2}$$

$$= \tfrac{1}{2}\left[\sin^{-1}\left(\tfrac{1}{2}\right) - \sin^{-1}0\right] = \frac{1}{2}\cdot\frac{\pi}{6} = \frac{\pi}{12}$$

EXAMPLE 9 Evaluate $\displaystyle\int \frac{1}{x^2+a^2}\,dx$.

SOLUTION To make the given integral more like Equation 13 we write

$$\int \frac{dx}{x^2+a^2} = \int \frac{dx}{a^2\left(\dfrac{x^2}{a^2}+1\right)} = \frac{1}{a^2}\int \frac{dx}{\left(\dfrac{x}{a}\right)^2+1}$$

This suggests that we substitute $u = x/a$. Then $du = dx/a$, $dx = a\,du$, and

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{a\,du}{u^2+1} = \frac{1}{a}\int \frac{du}{u^2+1} = \frac{1}{a}\tan^{-1}u + C$$

Thus, we have the formula

One of the main uses of inverse trigonometric functions is that they often arise when we integrate rational functions.

14
$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

EXAMPLE 10 Find $\displaystyle\int \frac{x}{x^4+9}\,dx$.

SOLUTION We substitute $u = x^2$ because then $du = 2x\,dx$ and we can use Equation 14 with $a = 3$:

$$\int \frac{x}{x^4+9}\,dx = \frac{1}{2}\int \frac{du}{u^2+9} = \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{x^2}{3}\right) + C$$

7.5 Exercises

1–10 Find the exact value of each expression.

1. (a) $\sin^{-1}(\sqrt{3}/2)$ (b) $\cos^{-1}(-1)$
2. (a) $\arctan(-1)$ (b) $\csc^{-1} 2$
3. (a) $\tan^{-1}\sqrt{3}$ (b) $\arcsin(-1/\sqrt{2})$
4. (a) $\sec^{-1}\sqrt{2}$ (b) $\arcsin 1$
5. (a) $\arccos(\cos 2\pi)$ (b) $\tan(\tan^{-1} 5)$
6. (a) $\tan^{-1}(\tan 3\pi/4)$ (b) $\cos(\arcsin \frac{1}{2})$
7. $\tan(\sin^{-1}(\frac{2}{3}))$
8. $\csc(\arccos \frac{3}{5})$
9. $\sin(2\tan^{-1}\sqrt{2})$
10. $\cos(\tan^{-1}2 + \tan^{-1}3)$

11. Prove that $\cos(\sin^{-1}x) = \sqrt{1-x^2}$ for $-1 \le x \le 1$.

12–14 Simplify the expression.

12. $\tan(\sin^{-1}x)$
13. $\sin(\tan^{-1}x)$
14. $\csc(\arctan 2x)$

15–16 Graph the given functions on the same screen. How are these graphs related?

15. $y = \sin x, \; -\pi/2 \leq x \leq \pi/2; \quad y = \sin^{-1}x; \quad y = x$

16. $y = \tan x, \; -\pi/2 < x < \pi/2; \quad y = \tan^{-1}x; \quad y = x$

17. Prove Formula 6 for the derivative of $\cos^{-1}$ by the same method as for Formula 3.

18. (a) Prove that $\sin^{-1}x + \cos^{-1}x = \pi/2$.
(b) Use part (a) to prove Formula 6.

19. Prove that $\dfrac{d}{dx}(\cot^{-1}x) = -\dfrac{1}{1 + x^2}$.

20. Prove that $\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{x\sqrt{x^2 - 1}}$.

21. Prove that $\dfrac{d}{dx}(\csc^{-1}x) = -\dfrac{1}{x\sqrt{x^2 - 1}}$.

22–35 Find the derivative of the function. Simplify where possible.

22. $y = \sqrt{\tan^{-1} x}$

23. $y = \tan^{-1}\sqrt{x}$

24. $h(x) = \sqrt{1 - x^2}\arcsin x$

25. $y = \sin^{-1}(2x + 1)$

26. $f(x) = x\ln(\arctan x)$

27. $H(x) = (1 + x^2)\arctan x$

28. $h(t) = e^{\sec^{-1}t}$

29. $y = \cos^{-1}(e^{2x})$

30. $y = x\cos^{-1}x - \sqrt{1 - x^2}$

31. $y = \arctan(\cos\theta)$

32. $y = \tan^{-1}(x - \sqrt{1 + x^2})$

33. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$

34. $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x - a}{x + a}}$

35. $y = \arccos\left(\dfrac{b + a\cos x}{a + b\cos x}\right), \quad 0 \leq x \leq \pi, \; a > b > 0$

36–37 Find the derivative of the function. Find the domains of the function and its derivative.

36. $f(x) = \arcsin(e^x)$

37. $g(x) = \cos^{-1}(3 - 2x)$

38. Find y' if $\tan^{-1}(xy) = 1 + x^2y$.

39. If $g(x) = x\sin^{-1}(x/4) + \sqrt{16 - x^2}$, find $g'(2)$.

40. Find an equation of the tangent line to the curve $y = 3\arccos(x/2)$ at the point $(1, \pi)$.

41–42 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

41. $f(x) = e^{-x}\arctan x$

42. $f(x) = x\arcsin(1 - x^2)$

43–46 Find the limit.

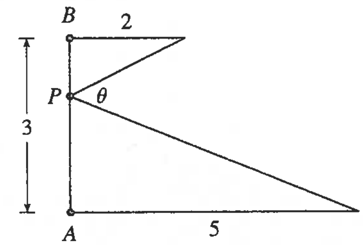
43. $\lim\limits_{x \to -1^+} \sin^{-1}x$

44. $\lim\limits_{x \to \infty} \arccos\left(\dfrac{1 + x^2}{1 + 2x^2}\right)$

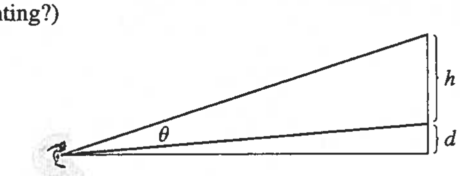
45. $\lim\limits_{x \to \infty} \arctan(e^x)$

46. $\lim\limits_{x \to 0^+} \tan^{-1}(\ln x)$

47. Where should the point P be chosen on the line segment AB so as to maximize the angle θ?



48. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at his eye by the painting?)



49. A ladder 10 ft long leans against a vertical wall. If the bottom of the ladder slides away from the base of the wall at a speed of 2 ft/s, how fast is the angle between the ladder and the wall changing when the bottom of the ladder is 6 ft from the base of the wall?

50. A lighthouse is located on a small island, 3 km away from the nearest point P on a straight shoreline, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

51–54 Sketch the curve using the guidelines of Section 4.5.

51. $y = \sin^{-1}\left(\dfrac{x}{x + 1}\right)$

52. $y = \tan^{-1}\left(\dfrac{x - 1}{x + 1}\right)$

53. $y = x - \tan^{-1}x$

54. $y = \tan^{-1}(\ln x)$

CAS **55.** If $f(x) = \arctan(\cos(3\arcsin x))$, use the graphs of f, f', and f'' to estimate the x-coordinates of the maximum and minimum points and inflection points of f.

56. Investigate the family of curves given by $f(x) = x - c\sin^{-1}x$. What happens to the number of maxima and minima as c changes? Graph several members of the family to illustrate what you discover.

57. Find the most general antiderivative of the function

$$f(x) = 2x + 5(1 - x^2)^{-1/2}$$

58. Find $f(x)$ if $f'(x) = 4 - 3(1 + x^2)^{-1}$ and $f(\pi/4) = 0$.

59–70 Evaluate the integral.

59. $\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1 - t^2}}\, dt$

60. $\int_0^1 \frac{4}{t^2 + 1}\, dt$

61. $\int_0^{\sqrt{3}/4} \frac{dx}{1 + 16x^2}$

62. $\int \frac{dt}{\sqrt{1 - 4t^2}}$

63. $\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1 - x^2}}\, dx$

64. $\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx$

65. $\int \frac{x + 9}{x^2 + 9}\, dx$

66. $\int \frac{\tan^{-1}x}{1 + x^2}\, dx$

67. $\int \frac{t^2}{\sqrt{1 - t^6}}\, dt$

68. $\int \frac{1}{x\sqrt{x^2 - 4}}\, dx$

69. $\int \frac{dx}{\sqrt{x}(1 + x)}$

70. $\int \frac{e^{2x}}{\sqrt{1 - e^{4x}}}\, dx$

71. Use the method of Example 9 to show that, if $a > 0$,

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

72. The region under the curve $y = 1/\sqrt{x^2 + 4}$ from $x = 0$ to $x = 2$ is rotated about the x-axis. Find the volume of the resulting solid.

73. Evaluate $\int_0^1 \sin^{-1}x\, dx$ by interpreting it as an area and integrating with respect to y instead of x.

74. Prove that, for $xy \neq 1$,

$$\arctan x + \arctan y = \arctan \frac{x + y}{1 - xy}$$

if the left side lies between $-\pi/2$ and $\pi/2$.

75. Use the result of Exercise 74 to prove the following:
(a) $\arctan \frac{1}{2} + \arctan \frac{1}{3} = \pi/4$
(b) $2 \arctan \frac{1}{3} + \arctan \frac{1}{7} = \pi/4$

76. (a) Sketch the graph of the function $f(x) = \sin(\sin^{-1}x)$.
(b) Sketch the graph of the function $g(x) = \sin^{-1}(\sin x)$, $x \in \mathbb{R}$.
(c) Show that $g'(x) = \frac{\cos x}{|\cos x|}$.
(d) Sketch the graph of $h(x) = \cos^{-1}(\sin x)$, $x \in \mathbb{R}$, and find its derivative.

77. Use the method of Example 7 to prove the identity

$$2 \sin^{-1}x = \cos^{-1}(1 - 2x^2) \qquad x \geq 0$$

78. Prove the identity

$$\arcsin \frac{x - 1}{x + 1} = 2 \arctan \sqrt{x} - \frac{\pi}{2}$$

79. Some authors define $y = \sec^{-1}x \iff \sec y = x$ and $y \in [0, \pi/2) \cup (\pi/2, \pi]$. Show that with this definition, we have (instead of the formula given in Exercise 20)

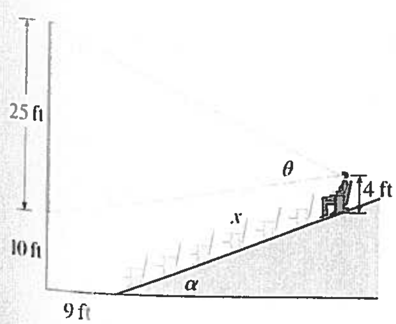
$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2 - 1}} \qquad |x| > 1$$

80. Let $f(x) = x \arctan(1/x)$ if $x \neq 0$ and $f(0) = 0$.
(a) Is f continuous at 0?
(b) Is f differentiable at 0?

APPLIED PROJECT

CAS Where to Sit at the Movies

A movie theater has a screen that is positioned 10 ft off the floor and is 25 ft high. The first row of seats is placed 9 ft from the screen and the rows are set 3 ft apart. The floor of the seating area is inclined at an angle of $\alpha = 20°$ above the horizontal and the distance up the incline that you sit is x. The theater has 21 rows of seats, so $0 \leq x \leq 60$. Suppose you decide that the best place to sit is in the row where the angle θ subtended by the screen at your eyes is a maximum. Let's also suppose that your eyes are 4 ft above the floor, as shown in the figure. (In Exercise 48 in Section 7.5 we looked at a simpler version of this problem, where the floor is horizontal, but this project involves a more complicated situation and requires technology.)



1. Show that

$$\theta = \arccos\left(\frac{a^2 + b^2 - 625}{2ab}\right)$$

where

$$a^2 = (9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2$$

and

$$b^2 = (9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2$$

61. $\displaystyle\int_4^6 \frac{1}{\sqrt{t^2 - 9}}\, dt$

62. $\displaystyle\int_0^1 \frac{dt}{\sqrt{16t^2 + 1}}\, dt$

63. $\displaystyle\int \frac{e^x}{1 - e^{2x}}\, dx$

64. Estimate the value of the number c such that the area under the curve $y = \sinh cx$ between $x = 0$ and $x = 1$ is equal to 1.

65. (a) Use Newton's method or a graphing device to find approximate solutions of the equation $\cosh 2x = 1 + \sinh x$.
(b) Estimate the area of the region bounded by the curves $y = \cosh 2x$ and $y = 1 + \sinh x$.

66. Show that the area of the shaded hyperbolic sector in Figure 6 is $A(t) = \frac{1}{2}t$. [*Hint:* First show that

$$A(t) = \tfrac{1}{2}\sinh t \cosh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$$

and then verify that $A'(t) = \frac{1}{2}$.]

67. Show that if $a \neq 0$ and $b \neq 0$, then there exist numbers α and β such that $ae^x + be^{-x}$ equals either $\alpha \sinh(x + \beta)$ or $\alpha \cosh(x + \beta)$. In other words, almost every function of the form $f(x) = ae^x + be^{-x}$ is a shifted and stretched hyperbolic sine or cosine function.

7.7 Indeterminate Forms and L'Hospital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although F is not defined when $x = 1$, we need to know how F behaves *near* 1. In particular, we would like to know the value of the limit

$$\boxed{1} \qquad \lim_{x \to 1} \frac{\ln x}{x - 1}$$

In computing this limit we can't apply Law 5 of limits (the limit of a quotient is the quotient of the limits, see Section 2.3) because the limit of the denominator is 0. In fact, although the limit in (1) exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type $\frac{0}{0}$**. We met some limits of this type in Chapter 2. For rational functions, we can cancel common factors:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x - 1)}{(x + 1)(x - 1)} = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}$$

We used a geometric argument to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

But these methods do not work for limits such as (1), so in this section we introduce a systematic method, known as *l'Hospital's Rule*, for the evaluation of indeterminate forms.

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of F and need to evaluate the limit

2 $$\lim_{x\to\infty} \frac{\ln x}{x-1}$$

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$. There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞; if the denominator wins, the answer will be 0. Or there may be some compromise, in which case the answer may be some finite positive number.

In general, if we have a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then the limit may or may not exist and is called an **indeterminate form of type** ∞/∞. We saw in Section 4.4 that this type of limit can be evaluated for certain functions, including rational functions, by dividing numerator and denominator by the highest power of x that occurs in the denominator. For instance,

$$\lim_{x\to\infty} \frac{x^2-1}{2x^2+1} = \lim_{x\to\infty} \frac{1-\dfrac{1}{x^2}}{2+\dfrac{1}{x^2}} = \frac{1-0}{2+0} = \frac{1}{2}$$

This method does not work for limits such as (2), but l'Hospital's Rule also applies to this type of indeterminate form.

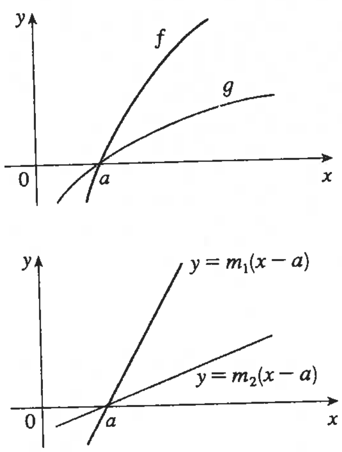


FIGURE 1

Figure 1 suggests visually why l'Hospital's Rule might be true. The first graph shows two differentiable functions f and g, each of which approaches 0 as $x \to a$. If we were to zoom in toward the point $(a, 0)$, the graphs would start to look almost linear. But if the functions actually *were* linear, as in the second graph, then their ratio would be

$$\frac{m_1(x-a)}{m_2(x-a)} = \frac{m_1}{m_2}$$

which is the ratio of their derivatives. This suggests that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

NOTE 1 ◦ L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.

NOTE 2 ◦ L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

NOTE 3 ◦ For the special case in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$, it is easy to see why l'Hospital's Rule is true. In fact, using the alternative form

L'Hospital's Rule is named after a French nobleman, the Marquis de l'Hospital (1661–1704), but was discovered by a Swiss mathematician, John Bernoulli (1667–1748). See Exercise 78 for the example that the Marquis used to illustrate his rule. See the project on page 504 for further historical details.

of the definition of a derivative, we have

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}=\frac{\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}$$

$$=\lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$=\lim_{x\to a}\frac{f(x)}{g(x)}$$

The general version of l'Hospital's Rule for the indeterminate form $\frac{0}{0}$ is somewhat more difficult and its proof is deferred to the end of this section. The proof for the indeterminate form ∞/∞ can be found in more advanced books.

EXAMPLE 1 Find $\displaystyle\lim_{x\to 1}\frac{\ln x}{x-1}$.

SOLUTION Since

$$\lim_{x\to 1}\ln x=\ln 1=0 \qquad\text{and}\qquad \lim_{x\to 1}(x-1)=0$$

we can apply l'Hospital's Rule:

Notice that when using l'Hospital's Rule we differentiate the numerator and denominator *separately*. We *do not* use the Quotient Rule.

$$\lim_{x\to 1}\frac{\ln x}{x-1}=\lim_{x\to 1}\frac{\dfrac{d}{dx}(\ln x)}{\dfrac{d}{dx}(x-1)}=\lim_{x\to 1}\frac{1/x}{1}$$

$$=\lim_{x\to 1}\frac{1}{x}=1$$

EXAMPLE 2 Calculate $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2}$.

The graph of the function of Example 2 is shown in Figure 2. We have noticed previously that exponential functions grow far more rapidly than power functions, so the result of Example 2 is not unexpected. See also Exercise 87.

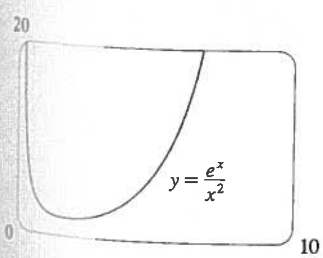


FIGURE 2

SOLUTION We have $\lim_{x\to\infty}e^x=\infty$ and $\lim_{x\to\infty}x^2=\infty$, so l'Hospital's Rule gives

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{\dfrac{d}{dx}(e^x)}{\dfrac{d}{dx}(x^2)}=\lim_{x\to\infty}\frac{e^x}{2x}$$

Since $e^x\to\infty$ and $2x\to\infty$ as $x\to\infty$, the limit on the right side is also indeterminate, but a second application of l'Hospital's Rule gives

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{e^x}{2x}=\lim_{x\to\infty}\frac{e^x}{2}=\infty$$

The graph of the function of Example 3 is shown in Figure 3. We have discussed previously the slow growth of logarithms, so it isn't surprising that this ratio approaches 0 as $x \to \infty$. See also Exercise 88.

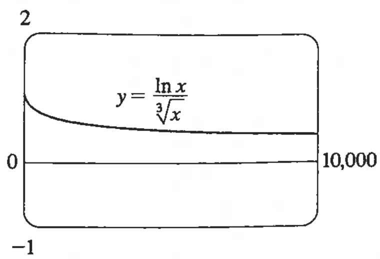


FIGURE 3

EXAMPLE 3 Calculate $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$.

SOLUTION Since $\ln x \to \infty$ and $\sqrt[3]{x} \to \infty$ as $x \to \infty$, l'Hospital's Rule applies:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0$$

EXAMPLE 4 Find $\lim_{x\to 0} \frac{\tan x - x}{x^3}$. [See Exercise 36(d) in Section 2.2.]

SOLUTION Noting that both $\tan x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we use l'Hospital's Rule:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply l'Hospital's Rule again:

$$\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}$$

The graph in Figure 4 gives visual confirmation of the result of Example 4. If we were to zoom in too far, however, we would get an inaccurate graph because $\tan x$ is close to x when x is small. See Exercise 36(d) in Section 2.2.

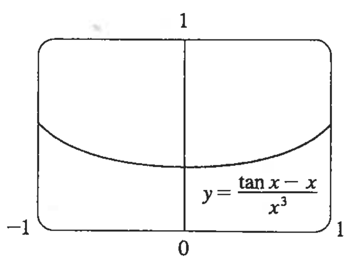


FIGURE 4

Because $\lim_{x\to 0} \sec^2 x = 1$, we simplify the calculation by writing

$$\lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x\to 0} \sec^2 x \lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x}$$

We can evaluate this last limit either by using l'Hospital's Rule a third time or by writing $\tan x$ as $(\sin x)/(\cos x)$ and making use of our knowledge of trigonometric limits. Putting together all the steps, we get

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}$$

$$= \frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x\to 0} \frac{\sec^2 x}{1} = \frac{1}{3}$$

EXAMPLE 5 Find $\lim_{x\to\pi^-} \frac{\sin x}{1 - \cos x}$.

SOLUTION If we blindly attempted to use l'Hospital's Rule, we would get

$$\lim_{x\to\pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x\to\pi^-} \frac{\cos x}{\sin x} = -\infty$$

This is *wrong*! Although the numerator $\sin x \to 0$ as $x \to \pi^-$, notice that the denominator $(1 - \cos x)$ does not approach 0, so l'Hospital's Rule can't be applied here.

The required limit is, in fact, easy to find because the function is continuous and the denominator is nonzero at π:

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1 - (-1)} = 0$$

Example 5 shows what can go wrong if you use l'Hospital's Rule without thinking. Other limits *can* be found using l'Hospital's Rule but are more easily found by other methods. (See Examples 3 and 5 in Section 2.3, Example 3 in Section 2.6, and the discussion at the beginning of this section.) So when evaluating any limit, you should consider other methods before using l'Hospital's Rule.

Indeterminate Products

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x \to a} f(x)g(x)$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type $0 \cdot \infty$**. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that we can use l'Hospital's Rule.

Figure 5 shows the graph of the function in Example 6. Notice that the function is undefined at $x = 0$; the graph approaches the origin but never quite reaches it.

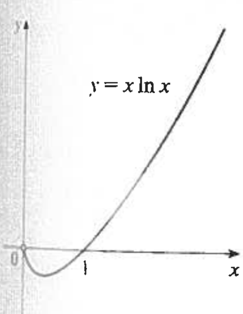


FIGURE 5

EXAMPLE 6 Evaluate $\lim_{x \to 0^+} x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

NOTE ◦ In solving Example 6 another possible option would have been to write

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{x}{1/\ln x}$$

This gives an indeterminate form of the type 0/0, but if we apply l'Hospital's Rule we get a more complicated expression than the one we started with. In general, when we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit.

EXAMPLE 7 Use l'Hospital's Rule to help sketch the graph of $f(x) = xe^x$.

SOLUTION Because both x and e^x become large as $x \to \infty$, we have $\lim_{x \to \infty} xe^x = \infty$. As $x \to -\infty$, however, $e^x \to 0$ and so we have an indeterminate product that requires the use of l'Hospital's Rule:

$$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = \lim_{x \to -\infty} (-e^x) = 0$$

Thus, the x-axis is a horizontal asymptote.

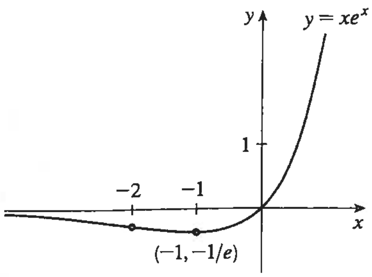


FIGURE 6

We use the methods of Chapter 4 to gather other information concerning the graph. The derivative is

$$f'(x) = xe^x + e^x = (x + 1)e^x$$

Since e^x is always positive, we see that $f'(x) > 0$ when $x + 1 > 0$, and $f'(x) < 0$ when $x + 1 < 0$. So f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$. Because $f'(-1) = 0$ and f changes from negative to positive at $x = -1$, $f(-1) = -e^{-1}$ is a local (and absolute) minimum. The second derivative is

$$f''(x) = (x + 1)e^x + e^x = (x + 2)e^x$$

Since $f''(x) > 0$ if $x > -2$ and $f''(x) < 0$ if $x < -2$, f is concave upward on $(-2, \infty)$ and concave downward on $(-\infty, -2)$. The inflection point is $(-2, -2e^{-2})$.

We use this information to sketch the curve in Figure 6.

Indeterminate Differences

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an **indeterminate form of type** $\infty - \infty$. Again there is a contest between f and g. Will the answer be ∞ (f wins) or will it be $-\infty$ (g wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.

EXAMPLE 8 Compute $\lim_{x\to(\pi/2)^-} (\sec x - \tan x)$.

SOLUTION First notice that $\sec x \to \infty$ and $\tan x \to \infty$ as $x \to (\pi/2)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\lim_{x\to(\pi/2)^-} (\sec x - \tan x) = \lim_{x\to(\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to(\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x\to(\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

Note that the use of l'Hospital's Rule is justified because $1 - \sin x \to 0$ and $\cos x \to 0$ as $x \to (\pi/2)^-$.

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x\to a} [f(x)]^{g(x)}$$

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0
2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

(Recall that both of these methods were used in differentiating such functions.) In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

EXAMPLE 9 Calculate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate. Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then

$$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x}$$

$$= \lim_{x \to 0^+} \frac{\dfrac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4$$

The graph of the function $y = x^x$, $x > 0$, is shown in Figure 7. Notice that although 0^0 is not defined, the values of the function approach 1 as $x \to 0^+$. This confirms the result of Example 10.

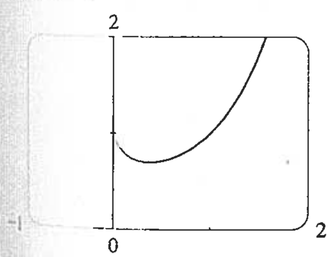


FIGURE 7

EXAMPLE 10 Find $\lim_{x \to 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. We could proceed as in Example 9 or by writing the function as an exponential:

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x \to 0^+} x \ln x = 0$$

Therefore

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1$$

See the biographical sketch of Cauchy on page 97.

In order to give the promised proof of l'Hospital's Rule we first need a generalization of the Mean Value Theorem. The following theorem is named after another French mathematician, Augustin-Louis Cauchy (1789–1857).

3 **Cauchy's Mean Value Theorem** Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Notice that if we take the special case in which $g(x) = x$, then $g'(c) = 1$ and Theorem 3 is just the ordinary Mean Value Theorem. Furthermore, Theorem 3 can be proved in a similar manner. You can verify that all we have to do is change the function h given by Equation 4.2.4 to the function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} [g(x) - g(a)]$$

and apply Rolle's Theorem as before.

Proof of L'Hospital's Rule We are assuming that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Let

$$L = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

We must show that $\lim_{x \to a} f(x)/g(x) = L$. Define

$$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases} \qquad G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

Then F is continuous on I since f is continuous on $\{x \in I \mid x \neq a\}$ and

$$\lim_{x \to a} F(x) = \lim_{x \to a} f(x) = 0 = F(a)$$

Likewise, G is continuous on I. Let $x \in I$ and $x > a$. Then F and G are continuous on $[a, x]$ and differentiable on (a, x) and $G' \neq 0$ there (since $F' = f'$ and $G' = g'$). Therefore, by Cauchy's Mean Value Theorem there is a number y such that $a < y < x$ and

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)}$$

Here we have used the fact that, by definition, $F(a) = 0$ and $G(a) = 0$. Now, if we let $x \to a^+$, then $y \to a^+$ (since $a < y < x$), so

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{F(x)}{G(x)} = \lim_{y \to a^+} \frac{F'(y)}{G'(y)} = \lim_{y \to a^+} \frac{f'(y)}{g'(y)} = L$$

A similar argument shows that the left-hand limit is also L. Therefore

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L$$

This proves l'Hospital's Rule for the case where a is finite.

If a is infinite, we let $t = 1/x$. Then $t \to 0^+$ as $x \to \infty$, so we have

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{t\to 0^+}\frac{f(1/t)}{g(1/t)}$$

$$= \lim_{t\to 0^+}\frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)} \qquad \text{(by l'Hospital's Rule for finite } a\text{)}$$

$$= \lim_{t\to 0^+}\frac{f'(1/t)}{g'(1/t)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

7.7 Exercises

1–4 Given that

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 1$$

$$\lim_{x\to a} p(x) = \infty \qquad \lim_{x\to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

1. (a) $\lim_{x\to a}\frac{f(x)}{g(x)}$ (b) $\lim_{x\to a}\frac{f(x)}{p(x)}$
(c) $\lim_{x\to a}\frac{h(x)}{p(x)}$ (d) $\lim_{x\to a}\frac{p(x)}{f(x)}$
(e) $\lim_{x\to a}\frac{p(x)}{q(x)}$

2. (a) $\lim_{x\to a}[f(x)p(x)]$ (b) $\lim_{x\to a}[h(x)p(x)]$
(c) $\lim_{x\to a}[p(x)q(x)]$

3. (a) $\lim_{x\to a}[f(x) - p(x)]$ (b) $\lim_{x\to a}[p(x) - q(x)]$
(c) $\lim_{x\to a}[p(x) + q(x)]$

4. (a) $\lim_{x\to a}[f(x)]^{g(x)}$ (b) $\lim_{x\to a}[f(x)]^{p(x)}$ (c) $\lim_{x\to a}[h(x)]^{p(x)}$
(d) $\lim_{x\to a}[p(x)]^{f(x)}$ (e) $\lim_{x\to a}[p(x)]^{q(x)}$ (f) $\lim_{x\to a}\sqrt[q(x)]{p(x)}$

5–62 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

5. $\lim_{x\to -1}\frac{x^2-1}{x+1}$

6. $\lim_{x\to -2}\frac{x+2}{x^2+3x+2}$

7. $\lim_{x\to 1}\frac{x^9-1}{x^5-1}$

8. $\lim_{x\to 1}\frac{x^a-1}{x^b-1}$

9. $\lim_{x\to(\pi/2)^+}\frac{\cos x}{1-\sin x}$

10. $\lim_{x\to 0}\frac{x+\tan x}{\sin x}$

11. $\lim_{t\to 0}\frac{e^t-1}{t^3}$

12. $\lim_{t\to 0}\frac{e^{3t}-1}{t}$

13. $\lim_{x\to 0}\frac{\tan px}{\tan qx}$

14. $\lim_{\theta\to\pi/2}\frac{1-\sin\theta}{\csc\theta}$

15. $\lim_{x\to\infty}\frac{\ln x}{x}$

16. $\lim_{x\to\infty}\frac{e^x}{x}$

17. $\lim_{x\to 0^+}\frac{\ln x}{x}$

18. $\lim_{x\to\infty}\frac{\ln\ln x}{x}$

19. $\lim_{t\to 0}\frac{5^t-3^t}{t}$

20. $\lim_{x\to 1}\frac{\ln x}{\sin\pi x}$

21. $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$

22. $\lim_{x\to 0}\frac{e^x-1-x-(x^2/2)}{x^3}$

23. $\lim_{x\to\infty}\frac{e^x}{x^3}$

24. $\lim_{x\to 0}\frac{\sin x}{\sinh x}$

25. $\lim_{x\to 0}\frac{\sin^{-1}x}{x}$

26. $\lim_{x\to 0}\frac{\sin x - x}{x^3}$

27. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

28. $\lim_{x\to\infty}\frac{(\ln x)^2}{x}$

29. $\lim_{x\to 0}\frac{x+\sin x}{x+\cos x}$

30. $\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}$

31. $\lim_{x\to\infty}\frac{x}{\ln(1+2e^x)}$

32. $\lim_{x\to 0}\frac{x}{\tan^{-1}(4x)}$

33. $\lim_{x\to 1}\frac{1-x+\ln x}{1+\cos\pi x}$

34. $\lim_{x\to\infty}\frac{\sqrt{x^2+2}}{\sqrt{2x^2+1}}$

35. $\lim_{x\to 1}\frac{x^a-ax+a-1}{(x-1)^2}$

36. $\lim_{x\to 0}\frac{1-e^{-2x}}{\sec x}$

37. $\lim_{x\to 0^+}\sqrt{x}\ln x$

38. $\lim_{x\to -\infty} x^2e^x$

39. $\lim_{x \to 0} \cot 2x \sin 6x$

40. $\lim_{x \to 0^+} \sin x \ln x$

41. $\lim_{x \to \infty} x^3 e^{-x^2}$

42. $\lim_{x \to \pi/4} (1 - \tan x) \sec x$

43. $\lim_{x \to 1^+} \ln x \tan(\pi x/2)$

44. $\lim_{x \to \infty} x \tan(1/x)$

45. $\lim_{x \to 0} \left(\frac{1}{x} - \csc x \right)$

46. $\lim_{x \to 0} (\csc x - \cot x)$

47. $\lim_{x \to \infty} \left(\sqrt{x^2 + x} - x \right)$

48. $\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x - 1} \right)$

49. $\lim_{x \to \infty} (x - \ln x)$

50. $\lim_{x \to \infty} (xe^{1/x} - x)$

51. $\lim_{x \to 0^+} x^{x^2}$

52. $\lim_{x \to 0^+} (\tan 2x)^x$

53. $\lim_{x \to 0} (1 - 2x)^{1/x}$

54. $\lim_{x \to \infty} \left(1 + \frac{a}{x} \right)^{bx}$

55. $\lim_{x \to \infty} \left(1 + \frac{3}{x} + \frac{5}{x^2} \right)^x$

56. $\lim_{x \to \infty} x^{(\ln 2)/(1 + \ln x)}$

57. $\lim_{x \to \infty} x^{1/x}$

58. $\lim_{x \to \infty} (e^x + x)^{1/x}$

59. $\lim_{x \to \infty} \left(\frac{x}{x + 1} \right)^x$

60. $\lim_{x \to 0} (\cos 3x)^{5/x}$

61. $\lim_{x \to 0^+} (\cos x)^{1/x^2}$

62. $\lim_{x \to \infty} \left(\frac{2x - 3}{2x + 5} \right)^{2x+1}$

63–64 Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

63. $\lim_{x \to \infty} x [\ln(x + 5) - \ln x]$

64. $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$

65–66 Illustrate l'Hospital's Rule by graphing both $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x = 0$ to see that these ratios have the same limit as $x \to 0$. Also calculate the exact value of the limit.

65. $f(x) = e^x - 1, \quad g(x) = x^3 + 4x$

66. $f(x) = 2x \sin x, \quad g(x) = \sec x - 1$

67–72 Use l'Hospital's Rule to help sketch the curve. Use the guidelines of Section 4.5.

67. $y = xe^{-x}$

68. $y = x(\ln x)^2$

69. $y = xe^{-x^2}$

70. $y = e^x/x$

71. $y = x - \ln(1 + x)$

72. $y = e^x - 3e^{-x} - 4x$

CAS **73–75**

(a) Graph the function.

(b) Use l'Hospital's Rule to explain the behavior as $x \to 0^+$ or as $x \to \infty$.

(c) Estimate the maximum and minimum values and then use calculus to find the exact values.

(d) Use a graph of f'' to estimate the x-coordinates of the inflection points.

73. $f(x) = x^{-x}$

74. $f(x) = (\sin x)^{\sin x}$

75. $f(x) = x^{1/x}$

76. Investigate the family of curves given by $f(x) = x^n e^{-x}$, where n is a positive integer. What features do these curves have in common? How do they differ from one another? In particular, what happens to the maximum and minimum points and inflection points as n increases? Illustrate by graphing several members of the family.

77. Investigate the family of curves given by $f(x) = xe^{-cx}$, where c is a real number. Start by computing the limits as $x \to \pm\infty$. Identify any transitional values of c where the basic shape changes. What happens to the maximum or minimum points and inflection points as c changes? Illustrate by graphing several members of the family.

78. The first appearance in print of l'Hospital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hospital in 1696. This was the first calculus *textbook* ever published and the example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. (At that time it was common to write aa instead of a^2.) Solve this problem.

79. If an initial amount A_0 of money is invested at an interest rate i compounded n times a year, the value of the investment after t years is

$$A = A_0 \left(1 + \frac{i}{n} \right)^{nt}$$

If we let $n \to \infty$, we refer to the *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after n years is

$$A = A_0 e^{it}$$

80. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

where g is the acceleration due to gravity and c is a positive constant. (In Chapter 10 we will be able to deduce this equation from the assumption that the air resistance is proportional to the speed of the object.)

(a) Calculate $\lim_{t\to\infty} v$. What is the meaning of this limit?

(b) For fixed t, use l'Hospital's Rule to calculate $\lim_{m\to\infty} v$. What can you conclude about the speed of a very heavy falling object?

81. In Section 5.3 we investigated the Fresnel function $S(x) = \int_0^x \sin(\frac{1}{2}\pi t^2)\,dt$, which arises in the study of the diffraction of light waves. Evaluate

$$\lim_{x\to 0} \frac{S(x)}{x^3}$$

82. Suppose that the temperature in a long thin rod placed along the x-axis is initially $C/(2a)$ if $|x| \leqslant a$ and 0 if $|x| > a$. It can be shown that if the heat diffusivity of the rod is k, then the temperature of the rod at the point x at time t is

$$T(x, t) = \frac{C}{a\sqrt{4\pi kt}} \int_0^a e^{-(x-u)^2/(4kt)}\,du$$

To find the temperature distribution that results from an initial hot spot concentrated at the origin, we need to compute

$$\lim_{a\to 0} T(x, t)$$

Use l'Hospital's Rule to find this limit.

83. If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x\to 0} \frac{f(2+3x) + f(2+5x)}{x}$$

84. For what values of a and b is the following equation true?

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0$$

85. If f' is continuous, use l'Hospital's Rule to show that

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x)$$

Explain the meaning of this equation with the aid of a diagram.

86. If f'' is continuous, show that

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x)$$

87. Prove that

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$$

for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.

88. Prove that

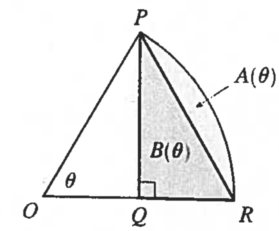
$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$$

for any number $p > 0$. This shows that the logarithmic function approaches ∞ more slowly than any power of x.

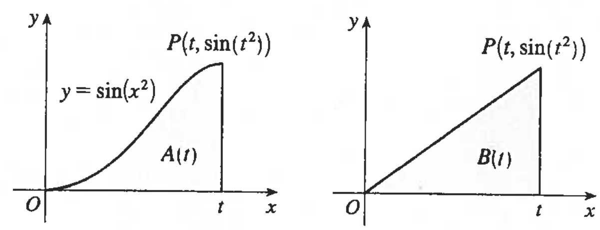
89. Prove that $\lim_{x\to 0^+} x^\alpha \ln x = 0$ for any $\alpha > 0$.

90. Evaluate $\displaystyle\lim_{x\to 0} \frac{1}{x^3}\int_0^x \sin(t^2)\,dt$.

91. The figure shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta\to 0^+} A(\theta)/B(\theta)$.



92. The figure shows two regions in the first quadrant: $A(t)$ is the area under the curve $y = \sin(x^2)$ from 0 to t, and $B(t)$ is the area of the triangle with vertices O, P, and $(t, 0)$. Find $\lim_{t\to 0^+} A(t)/B(t)$.



93. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Use the definition of derivative to compute $f'(0)$.

(b) Show that f has derivatives of all orders that are defined on $\mathbb{R}$. [*Hint:* First show by induction that there is a polynomial $p_n(x)$ and a nonnegative integer k_n such that $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$.]

94. Let

$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(a) Show that f is continuous at 0.

(b) Investigate graphically whether f is differentiable at 0 by zooming in several times toward the point (0, 1) on the graph of f.

(c) Show that f is not differentiable at 0. How can you reconcile this fact with the appearance of the graphs in part (b)?