Transmission Radiography and Tomography

A Simplified Overview

This material provides a brief overview of radiographic principles prerequisite to Lab #2 of the Radiography and Tomography Linear Algebra Modules. The goal is to develop the basic discrete radiographic operator for axial tomography of the human body.

What is Radiography?

Transmission radiography and tomography are familiar and common processes in today's world, especially in medicine and non-destructive testing in industry. Some examples include

- Single-view X-ray radiography is used routinely to view inside the human body; for example, bone fracture assessment, mammography, and angiographic procedures.
- Multiple-view X-ray radiography is realized in computerized axial tomography (CAT) scans used to provide 3D images of body tissues.
- Neutron and X-ray imaging is used in industry to quantify manufactured part assemblies or defects which cannot be visually inspected.

Transmission Radiography is the process of measuring and recording changes in a high-energy particle beam (X-rays, protons, neutrons, etc.) resulting from passage through an object of interest.

Tomography is the process of infering properties of an unknown object by interpreting radiographs of the object.

X-rays, just like visible light, are photons or electromagnetic radiation, but at much higher energies and outside of the range of our vision. Because of the wavelength of typical X-rays (on the order of a nanometer), they readily interact with objects of similar size such as individual molecules or atoms. This property makes them particularly useful in transmission imaging. Figure 1 is a cartoon of a typical x-ray radiographic experiment or procedure. An x-ray beam is produced with known energy and geometric characteristics. The beam is aimed at a region of interest. The photons interact with matter in the region of interest, changing the intensity, energy and geometry of the beam. A detector measures the pattern (and possibly the distribution) of incident energy. The detection data, when compared to the incident beam characteristics, contains the known signature of the region of interest. We consider the mathematics and some of the physics involved in each step with the goal of modeling a radiographic transformation appropriate for mixed soft and hard tissue axial tomography of the human body.

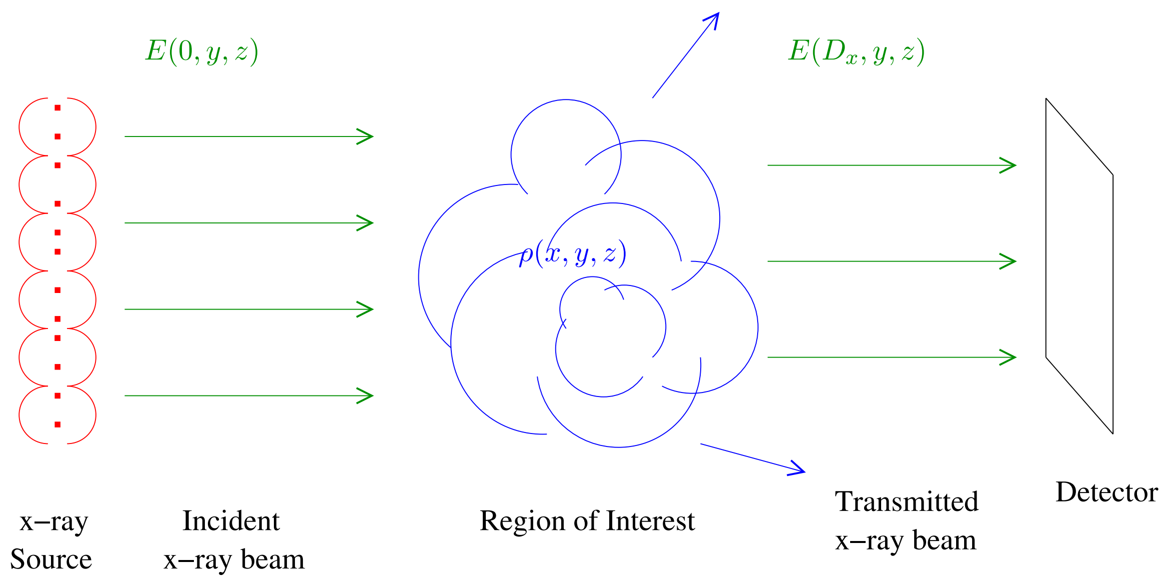

Figure 1: Typical radiographic experiment.

The Incident X-ray Beam

We begin with an x-ray beam in which the x-ray photons all travel parallel to each other in the beam direction, which we take to be the positive x-direction. Additionally we assume that the beam is of short time duration, the photons being clustered in a short pulse instead of being continuously produced. A beam with these geometric characteristics is usually not directly acheivable through typical x-ray sources (see supplementary material for some discussion on x-ray sources). However, such a beam can be approximated readily through so-called collimation techniques which physically limit the incident x-rays to a subset that compose a planar (neither convergent nor divergent) beam.

While not entirely necessary for the present formulation, we consider a monochromatic x-ray source. This means that every x-ray photon produced by the source has exactly the same "color." The term "monochromatic" comes from the visible light analog in which, for example, a laser pointer may produce photons of only one color, red. The energy of a single photon is proportional to its frequency ν, or inversely proportional to its wavelength λ. The frequency, or wavelength, determine the color, in exact analogy with visible light. In particular, the energy of a single photon is $h\nu$ where the constant of proportionality h is known as Planck's constant.

The intensity (or brightness), $E(x, y, z)$, of a beam is proportional to the photon density. The intensity of the beam just as it enters the region of interest at $x = 0$ is assumed to be the same as the itensity at the source. We write both as $E(0, y, z)$. It is assumed that this quantity is well known or independently measureable.

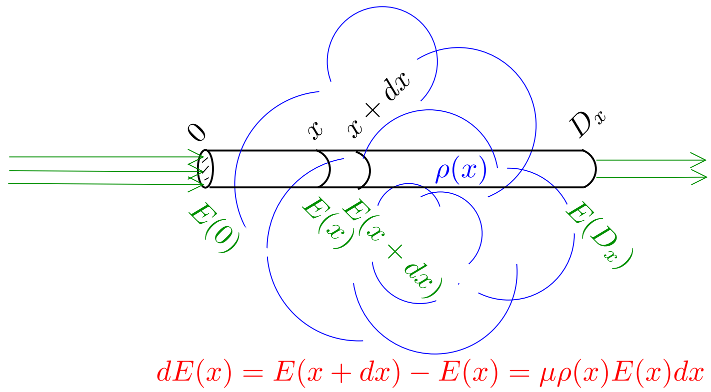

Figure 2: X-ray beam attenuation computation for a beam path of fixed y and z.

X-Ray Beam Attenuation

As the beam traverses the region of interest, from $x = 0$ to $x = D_x$ (see Figure 2), the intensity changes as a result of interactions with matter. In particular, a transmitted (resultant) intensity $E(D_x, y, z)$ exits the far side of the region of interest. We will see that under reasonable assumptions this transmitted intensity is also planar but is reduced in magnitude. This process of intensity reduction is called attenuation. It is our goal in this section to model the attenuation process.

X-ray attenuation is a complex process that is a combination of several physical mechanisms (outlined in the supplementary material) describing both scattering and absorption of x-rays. We will consider only Compton scattering which is the dominant process for typical medical x-ray radiography. In this case, attenuation is almost entirely a function of photon energy and material mass density. As we are considering monochromatic (monoenergetic) photons, attenuation is modeled as a function of mass density only.

Consider the beam of initial intensity $E(0, y, z)$ passing through the region of interest, at fixed y and z. The relative intensity change in an infinitesimal distance from x to $x + dx$ is proportional to the mass density $\rho(x, y, z)$ and is given by

$$dE(x, y, z) = E(x + dx, y, z) - E(x, y, z) = -\mu\rho(x, y, z)E(x, y, z)dx$$

where μ is a factor that is nearly constant for many materials of interest. We also assume that any scattered photons exit the region of interest without further interaction. This is the so-called single-scatter approximation which dictates that the intensity remains planar for all x.

Integrating over the path from $x = 0$ where the initial beam intensity is $E(0, y, z)$ to $x = D_x$ where the beam intensity is $E(D_x, y, z)$ yields

$$\frac{dE(x, y, z)}{E(x, y, z)} = -\mu\rho(x, y, z)dx$$

$$\int_{E(0,y,z)}^{E(D_x,y,z)} \frac{dE(x, y, z)}{E(x, y, z)} = -\mu \int_0^{D_x} \rho(x, y, z)dx$$

$$\ln E(D_x, y, z) - \ln E(0, y, z) = -\mu \int_0^{D_x} \rho(x, y, z) dx$$

$$E(D_x, y, z) = E(0, y, z) e^{-\mu \int_0^{D_x} \rho(x,y,z) dx}.$$

This expression shows us how the initial intensity is reduced, because photons have been scattered out of the beam. The relative reduction depends on the density (or mass) distribution in the region of interest.

Radiographic Energy Detection

The transmitted intensity $E(D_x, y, z)$ continues to travel on to a detector (e.g. film) which records the total detected energy in each of m detector bins. The detected energy in any bin is the intensity integrated over the bin cross-sectional area. Let p_k be the number of x-ray photons collected at detector bin k. p_k is then the collected intensity integrated over the bin area and divided by the photon energy.

$$p_k = \frac{1}{h\nu} \iint_{(bin\ k)} E(0, y, z) \left(e^{-\mu \int_0^{D_x} \rho(x,y,z) dx} \right) dydz.$$

Let the bin cross sectional area, σ, be small enough so that both the contributions of the density and intensity to the bin area integration are approximately a function of x only. Then

$$p_k = \frac{\sigma E(0, y_k, z_k)}{h\nu} e^{-\mu \int_0^{D_x} \rho(x, y_k, z_k) dx},$$

where y_k and z_k locate the center of bin k. Let p_k^0 be the number of x-ray photons initially aimed at bin k, $p_k^0 = \sigma E(0, x, y)/h\nu$. Due to attenuation, $p_k \leq p_k^0$ for each bin.

$$p_k = p_k^0 e^{-\mu \int_0^{D_x} \rho(x, y_k, z_k) dx}.$$

Equivalently, we can write (multiply the exponent argument by σ/σ):

$$p_k = p_k^0 e^{-\frac{\mu}{\sigma} \int_0^{D_x} \sigma \rho(x, y_k, z_k) dx}.$$

The remaining integral is the total mass in the region of interest that the x-ray beam passes through to get to bin k. We will call this mass s_k. Now we have

$$p_k = p_k^0 e^{-s_k/\alpha},$$

where $\alpha = \sigma/\mu$. This expression tells us that the number of photons in the part of the beam directed at bin k is reduced by a factor that is exponential in the total mass encountered by the photons.

Finally, we note that the detector bins correspond precisely to pixels in a radiographic image.

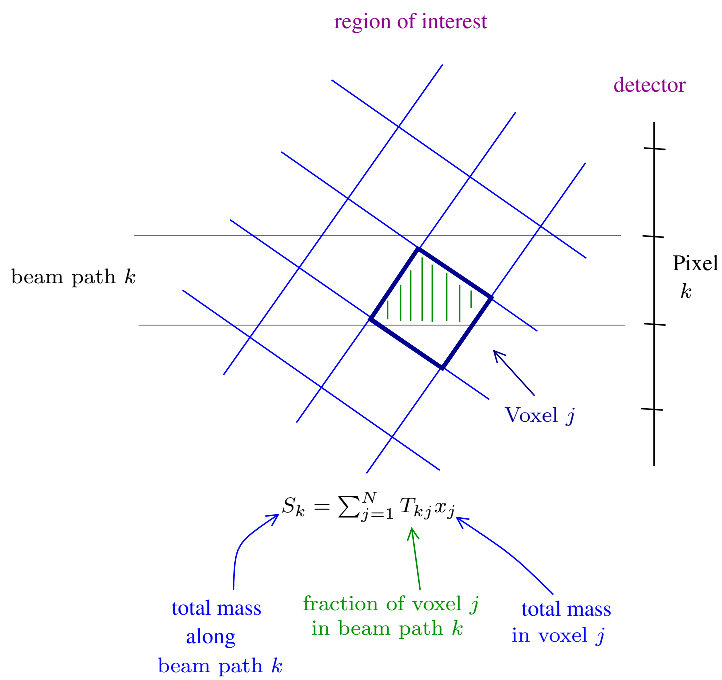

Figure 3: Object space and radiograph space discretization.

The Radiographic Transformation Operator

We consider a region of interest subdivided into N cubic voxels (three-dimensional pixels). Let x_j be the mass in object voxel j and T_{kj} the fraction of voxel j in beam path k (see Figure 3). Then the mass along beam path k is

$$s_k = \sum_{j=1}^{N} T_{kj} x_j,$$

and the expected photon count at radiograph pixel k, p_k, is given by

$$p_k = p_k^0 e^{-\frac{1}{\alpha} \sum_{j=1}^{N} T_{kj} x_j},$$

or equivalently,

$$b_k \equiv \left(-\alpha \ln \frac{p_k}{p_k^0} \right) = \sum_{j=1}^{N} T_{kj} x_j.$$

The new quantities b_k represent a variable change that allows us to formulate the matrix expression for the radiographic transformation

$$b = Tx.$$

This expression tells us that given a voxelized object mass distribution image $x \in \mathbb{R}^N$, the expected radiographic data (mass projection) is image $b \in \mathbb{R}^m$, with the two connected

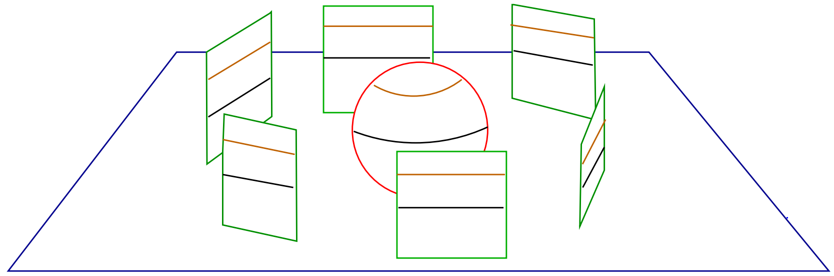

Figure 4: Example axial radiography scenario with six equally spaced views. Horizontal slices of the object project to horizontal rows in the radiographs.

through radiographic transformation $T \in \mathcal{M}_{m\times N}(\mathbb{R})$. The mass projection b and actual photon counts p and p^0 are related as given above. It is important to note that b_k is defined only for $p_k > 0$. Thus, this formulation is only valid for radiographic scenarios in which every radiograph detector pixel records at least one hit. This is always the case for medical applications which require high constrast and high signal-to-noise ratio data.

Multiple Views and Axial Tomography

Thus far, we have a model that can be used to compute a single radiograph of the region of interest. In many applications it is beneficial to obtain several or many different views of this region. In some industrial applications, the region of interest can be rotated within the radiographic apparatus, with radiographs obtained at various rotation angles. In medical applications the radiographic apparatus is rotated about the region of interest (including the subject!). In this latter case, the voxelization remains fixed and the coordinate system rotates. For each of a view angles the new m pixel locations require calculation of new mass projections T_{kj}. The full multiple-view operator contains mass projections onto all $M = a\cdot m$ pixel locations. Thus, for multiple-view radiography: x is still a vector in $\mathbb{R}^N$, but b is a vector in $\mathbb{R}^M$ and T is a matrix operator in $\mathcal{M}_{M\times N}(\mathbb{R})$.

Finally, we make the distinction between the general scenario and axial tomography (CAT scans). In principle, we could obtain radiographs of the region of interest from any direction (above, below, left, right, front, back, etc.). However, in axial tomography the physical limitations of the apparatus and subject placement dictate that views from some directions are not practical. The simplest scenario is to obtain multiple views by rotating the apparatus about a fixed direction perpendicular to the beam direction. This is why CAT machines have a donut or tube shaped appearance within which the apparatus is rotated. The central table allows the subject to rest along the rotation axis and the beam can pass through the subject along trajectories.

This axial setup also simplifies the projection operator. If we consider the ℓ^{th} slice of the region of interest, described by an $n \times n$ arrray of N voxels, the mass projections of this slice will only occur in the ℓ^{th} row of pixels in each radiographic view see Figure 4. As a result, 3D reconsructions can be obtained by a series of independent 2D reconstructed slices. For example, the brown slice of the spherical object (represented in $\mathbb{R}^N$) is related to the collection of brown rows of the radiographs (represented in $\mathbb{R}^M$) through the projection operator $T \in \mathcal{M}_{M\times N}(\mathbb{R})$. The black slice and black rows are related through the *same* projection operator.

Model Summary

The list below gathers the various mathematical quantities of interest.

- N is the number of object voxels.
- M is the number of radiograph pixels.
- $x \in \mathbb{R}^N$ is the material mass in each object voxel.
- $b \in \mathbb{R}^M$ is the mass projection onto each radiograph pixel.
- $p \in \mathbb{R}^M$ is the photon count recorded at each radiograph pixel.
- $T \in \mathcal{M}_{N\times M}(\mathbb{R})$ is voxel volume projection operator. T_{ij} is the fractional volume of voxel j which projects orthogonally onto pixel i.
- p_k^0 is the incident photon count per radiograph pixel.
- $b = -\alpha \ln \frac{p}{p^0}$.
- $b = Tx$ is the (mass projection) radiographic transformation.

The description of images (objects and radiographs) as vectors in $\mathbb{R}^N$ and $\mathbb{R}^M$ is computationally useful and more familiar than a vector spaces of images. One should keep in mind that this is a particular representation for images which is useful as a tool but is not geometrically descriptive. The price we pay for this convenience is that we no longer have the geometry of the radiographic setup (pixelization and voxelization) encoded in the representation.

A vector in $\mathbb{R}^3$, say $(1, 2, 5)$, is a point in a three-dimensional space with coordinates described relative to three orthogonal axes. We can actually locate this point and plot it. An image represented in $\mathbb{R}^3$, say $(1, 2, 5)$, is *not* a point in this space. Without further information about the vector space of which it is a member, we cannot draw this image. The use of $\mathbb{R}^3$ allows us to perform scalar multiplication and vector addition on images *because these operations are equivalently defined on* $\mathbb{R}^3$.

Model Assumptions

The radiography model we have constructed is based on a number of approximations and assumptions which we list here. This list is not comprehensive, but it does gather the most important concepts.

- **Monochromaticity.** Laboratory sources generate x-rays with a wide range of energies as a continuous spectrum. We say that such x-ray beams are polychromatic. One method of approximating a monochromatic beam is to precondition the beam by having the x-rays pass through a uniform material prior to reaching the region of interest. This process preferentially attenuates the lower energy photons, leaving only the highest energy photons. This process is known as beam-hardening. The result is a polychromatic beam with a narrower range of energies. We can consider the beam to be approximately monochromatic, especially if the attenuation coefficient(s) of the material, μ, is not a strong function of photon energy.

- **Geometric Beam Characteristics.** Laboratory sources do not naturally generate planar x-ray beams. It is more characteristic to have an approximate point source with an intensity pattern that is strongly directionally dependent. Approximate planar beams with relatively uniform intensity $E(0, y, x)$ can be achieved by selective beam shielding and separation of source and region of interest. In practice, it is common to use the known point source or line source characteristics instead of assuming a planar beam. The model described here is unchanged except for the computation of T itself.

- **Secondary Radiation.** Our model uses a single-scatter approximation in which if a photon undergoes a Compton scatter, it is removed from the analysis. In fact, x-rays can experience multiple scatter events as they traverse the region of interest or other incidental matter (such as the supporting machinery). The problematic photons are those that scatter one or more times *and* reach the detector. This important secondary effect is often approximated by more advanced models.

- **Energy-Dependent Attenuation.** The attenuation coefficient μ, which we have taked to be constant, is not only somewhat material dependent but is also beam energy dependent. If the beam is truly monochromatic this is not a problem. However, for a polychromatic beam the transmitted total energy will depend on the *distribution* of mass along a path, not just the total mass.

- **Other Attenuation Mechanisms.** We have included only Compton scattering in the model. Four other mechanisms (outlined in the supplementary material) contribute to the attenuation. While Compton scattering is the dominant contributor, photoelectric scattering will have some effect. It becomes important at the lower energies of interest and for materials of relatively high atomic number – such as calcium which is concentrated in bone. The major effect of ignoring photoelectric scattering is quantitative mass uncertainty.

- There are a number of **Detector-Related Effects** which affect radiograph accuracy. Energy detection efficiency can be a function of beam intensity, photon energy and even pixel location. Detectors are subject to point-spread effects in which even an infinitesimally narrow beam results in a finitely narrow detection spot. These types of effects are usually well-understood, documented and can be corrected for in the data. Detection is also prone to noise from secondary radiation, background radiation, or simply manufacturing variances.

Additional Resources

There is a variety of online source material that expands on any of the material presented here. Here are a few starting points.

https://www.nde-ed.org/EducationResources/CommunityCollege/Radiography/c_rad_index.htm

http://web.stanford.edu/group/glam/xlab/MatSci162_172/LectureNotes/01_Properties%20&%20Safety.pdf

http://radiologymasterclass.co.uk/tutorials/physics/x-ray_physics_production.html

Several physical processes contribute to absorption and scattering of individual photons as they pass through matter. Collectively, these processes alter the geometry and intensity of a beam of such photons. What follows is a brief description of each.

- The **Photoelectric Effect** is the absorption of an x-ray photon by an atom accompanied by the ejection of an outer-shell electron. This ionized atom then re-absorbs an electron and emits an x-ray of energy characteristic of the atom. The daughter x-ray is a low-energy photon which is quickly re-absorbed and is effectively removed from the x-ray beam. Photoelectric absorption is the dominant process for photon energies below about 100keV and when interacting with materials of high atomic number.

- **Rayleigh Scattering** is the process of a photon interacting with an atom without energy loss. The process is similar to the collision of two billiard balls. Rayleigh scattering is never the dominant mechanism, at any energy.

- **Compton Scattering** occurs when an x-ray photon interacts with an electron imparting some energy to the electron. Both electron and photon are emitted and the photon undrgoes a directional change or scatter. Compton Scattering is the dominant process for soft tissue at photon energies between about 100keV through about 8MeV.

- **Pair Production** is the process in which a photon is absorbed producing a positron-electron pair. The positron quickly decays into two 510keV x-ray photons. Pari production is only significant for photon energies of several MeV or more.

- **Photodisintegration** can occur for photons of very high energy. Photodisintegration is the process of absorption of the photon by an atomic nucleus and the subsequent ejection of a nuclear particle.