Radiograpy and Tomography in Linear Algebra
Lab #5: Reconstruction Without an Inverse

In this lab, we will consider the cases when the matrix representation of the radiographic transformation does not have an inverse. (Note: we say that $T : V \to W$ has an **inverse** $T^{-1} : W \to V$ if $T^{-1}T = I_V$, the identity mapping from V to V, and $TT^{-1} = I_W$ the identity mapping from W to W. We say that $T : V \to W$ has a **one-sided inverse**, $P : W \to V$ if $PT = I_V$.)

Invertible transformation

In this section, we will consider the following example: We are given a radiograph with 24 pixels that was created by applying some radiographic transformation, T to an object with 16 voxels.

1. Give a scenario for a radiographic transformation T that fits the above example. Don't calculate a T, rather give the following:

   - Size of the object: ____ $\times$ ____.
   - Number of pixels per view:
   - Number of views:

2. Suppose we know b and we want to find x. This means that we want

$$x = T^{-1}b.$$

   (a) What properties must the *transformation* T have so that T is invertible?

   (b) What properties must the *transformation* T have so that the one-sided inverse of T exists?

   (c) What matrix properties must the *matrix representation* of T have so that it is invertible?

   (d) When $N \leq M$ (as in the example above), what matrix properties must the *matrix representation* of T have so that it has a one-sided inverse?

3. For ease of notation, we typically use the same notation for the matrix and the transformation, that is, we call the matrix representation of T, T. Suppose, $N \leq M$ and a one-sided inverse, P of T exists. This means that $x = Pb$. We know that if T is invertible, we have that $P = T^{-1}$ and $x = Pb$. But, in the example above, we know that T is not invertible. Using the following steps, find the one-sided inverse of T.

   (a) Because $Tx = b$, for any linear operator A, we can write $ATx = Ab$. This is helpful if AT is invertible. Since T is one-to-one, we know that for AT to be invertible, the only vector in $\text{ran}(T)$ that is in $\text{null}(A)$ is the zero vector. What other properties must A have so that AT is invertible?

(b) Provide a matrix, A so that A has the properties you listed in 3a and so that AT is invertible.

(c) Solve for x in the matrix equation $ATx = Ab$ using the A you found and provide a representation of the one-sided inverse of P.

4. Putting this all together now, state the necessary and sufficient condition for T to have a one-sided inverse?

Brain scan reconstructions

For each of the scenarios you are about to see, write down the details of the scenario (number of views and size of T), the outcome of finding a one-sided inverse (no one-sided inverse exists or an one-sided inverse exists), and the reason this happens.

1.

2.

3.

4.

5.

6.

7.

But wait! We have the video from the beginning of class! This reconstruction, on a 108×108 object used only 30 views with 108 pixels per view. Does a one-sided inverse exist?

Application to a small example

Consider the following radiographic example.

- Total number of voxels: $N = 16$ $(n = 4)$.
- Total number of pixels: $M = 24$
- $ScaleFac = 1$
- Number of views: $a = 6$
- View angles: $\theta_1 = 0°$, $\theta_2 = 20°$,$\theta_3 = 40°$,$\theta_4 = 60°$,$\theta_5 = 80°$,$\theta_6 = 100°$.

1. Use tomomap.m to compute T and verify that the one-sided inverse of T must exist.

2. Compute the one-sided inverse P. Use P to find the object that created the following radiograph vector (You should be able copy and paste this into Octave or Matlab. If you need assistance in using your one-sided inverse, ask and I will help you with syntax):

```
b=[0.00000
   32.00000
   32.00000
    0.00000
    1.97552
   30.02448
   30.02448
    1.97552
    2.71552
   29.28448
   29.28448
    2.71552
    2.47520
   29.52480
   29.52480
    2.47520
    1.17456
   30.82544
   30.82544
    1.17456
    1.17456
   30.82544
   30.82544
    1.17456]
```