name:____________________________________

MATH 256 – FINAL EXAM

<u>Directions:</u> You may use *your own* textbook but **<u>not</u>** any solutions manual. You may use **your own** notes and **your own** homework. **You may not use the internet or any other electronic or printed resources including mathematical software, or other books.** **<u>You may not use anyone elses notes</u>**. If you work on this exam in a public space you must keep your work private. Do not leave your work where it is visible. Erase boards if you work on them.

<u>You may not discuss this exam with or near any person except Dr. Heather Moon.</u> Failure to follow the letter <u>and spirit</u> of these instructions will result in you failing the course. Your solutions should be **written very clearly and clean**. Your audience is not Heather, nor is it Morgan. Write something your classmates will understand. You must provide sufficient reasoning to completely convince any of your classmates.

The exam is due no later than **<u>5:00 PM on Tuesday, December 15</u>**. You must turn this page in, signed in order to get any credit on this exam.

I have not received any help from a source that would violate the spirit of the above directions.

signed:____

(1) Given the set $V = \left\{ \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \middle| \, a+b-c+2d=0, 2a+b=0, c-d=0, a,b,c,d,e,f \in \mathbb{R} \right\}$.

   (a) Show V is a vector space.
   (b) Find a basis for V and show that it is indeed a basis.
   (c) What is $\dim V$?

(2) Using V above and given the transformation $T : V \to W$ defined by

$$T \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = (a+e)x^2 + (b-f)x + (c+f).$$

   (a) Show that T is linear.
   (b) State the domain and codomain of T.
   (c) Find $\operatorname{ran}(T)$
   (d) Find $\operatorname{null}(T)$
   (e) Find $\operatorname{rank}(T)$
   (f) Find $\operatorname{nullity}(T)$
   (g) Is T 1-1? If so, prove it. If not, prove it.
   (h) Is T onto? If so, prove it. If not, prove it.

(3) Now, define $T : \mathcal{M}_{3\times 2} \to \mathcal{P}_4$ by

$$T \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = ax^4 + (b+d)x^3 + (c+e+f).$$

(Note: This transformation has nothing to do with the above transformation or vector spaces.)

   (a) Find a matrix representation for T using the standard bases for $\mathcal{M}_{3\times 2}$ and $\mathcal{P}_4$.
   (b) Find another basis for $\mathcal{M}_{3\times 2}$, call it $\mathcal{B}$. Find the matrix representation for the change of basis from the standard basis to $\mathcal{B}$.
   (c) Given $v = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{pmatrix}$. Find $[v]_{\mathcal{B}}$.

(4) Given the matrix

$$M = \begin{pmatrix} 0 & 1 & 1 \\ -3 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix}$$

(a) Find the eigenvalues and corresponding eigenspaces of M.

(b) Diagonalize M or give a reason why you know M is not diagonalizeable.

(5) Suppose you model a diffusion process defined by $u(t + \Delta t) = Mu(t)$ for all time t and some fixed Δt and for $u \in \mathbb{R}^4$. Suppose that M is a 4×4 matrix with eigenvalues $\lambda_1 = 1, \lambda_2 = .5, \lambda_3 = .25$, and $\lambda_4 = .125$ and corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \text{and} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \text{respectively.}$$

(a) Suppose you start with an initial heat state of $u(0) = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}$, what is $u(\Delta t)$?

(b) What is $u(100\Delta t)$? (You need not compute this exactly, rather write it in a way that makes the next question easier to answer.)

(c) What is the long term behavior of this diffusion? That is, what is $\lim_{t\to\infty} u(t)$?