Heat Equation for Linear Algebra
Lab #3

In this lab, you will explore details of the nature of heat diffusion. This lab requires that you use Matlab or Octave and gives you commands to type at the Matlab/Octave prompt.

1. First watch the demo from class of the diffusion on a heat state by typing the following command in Matlab/Octave:

   ```
   HeatEqnClassDemos(1);
   ```

   What characteristics of heat flow (discussed at the beginning of this module) did you observe in the demo? In the rest of this lab, you'll discover how those characteristics can be traced back to linear algebra concepts that we've been studying!

2. First start by finding the eigenvectors and eigenvalues for the heat diffusion operator. Do this using the `eig` command in MatLab (or Octave). First, run the function that creates the diffusion operator by typing, at the Matlab/Octave prompt, the two lines below (with a "return" or "enter" key).

   ```
   m=5;
   E=full(EvolutionMatrix(m));
   ```

   Check that E looks like you want it to by typing

   ```
   E
   ```

   What is the value of δ used by this code? Now, use then use the `eig` command to find the eigenvalues and eigenvectors

   ```
   [V,D]=eig(E);
   ```

3. Now, we want to verify that V is the matrix whose columns are eigenvectors and D is the matrix whose diagonal entries are the eigenvalues.

   (a) To see that D is actually diagonal, type

   ```
   D
   ```

   (b) Now verify that the first column of V is the eigenvector of E whose eigenvalue is the first diagonal entry of D. Do this by typing in Matlab/Octave

```
E*V(:,1)
D(1,1)*V(:,1)
```

The first of these commands multiplies our diffusion operator E by the first column of the V matrix. The second of these commands multiplies the $(1,1)$ entry of the D matrix by the first column of V. These should be the same.

(c) Try this again with the second column of V and the $(2,2)$ entry of D. You may notice that the third entry in this eigenvector may be represented by a very small value $\sim 10^{-16}$. This is a numerical artifact; such small values in relation to other entries should be taken to be zero.

(d) Now we can get a better form for the entries in D. Type

```
L=diag(D)
```

This gives a vector made of the diagonal elements of D. (Caution: the `diag` command has many uses other than extracting diagonal elements.)

4. Let's see what this has to do with our heat diffusion.

   (a) Now redo the steps (2) and (3d) in Matlab/Octave with `m=100` to get the new eigenvectors and eigenvalues of E.

   (b) Now we will visualize these eigenvectors as heat states.

      i. We will choose to view 5 eigenvectors with their the corresponding eigenvalues. Use the plotting function `EigenStuffPlot` by typing

      ```
      choices=[80,85,90,95,100];
      EigenStuffPlot(V(:,choices),L(choices));
      ```

      ii. How are the individual eigenvectors similar or dissimilar? Make some observations about the relationship between these eigenvectors and eigenvalues.

      iii. Choose different eigenvectors (remember we have 100 of them) to plot by changing entries in vector in the command that starts

      ```
      choices=...
      ```

      What did you choose for `choices`?

      iv. Now rerun the plotting function by entering

      ```
      EigenStuffPlot(V(:,choices),L(choices));
      ```

      v. Write a list of observations relating eigenvectors and eigenvalues. (Try the above commands with a few more choices if needed to answer this question.)

5. In the previous lab, you wrote an arbitrary heat state as a linear combination of the eigenvectors. Let's view some diffusions of linear combinations of a few eigenvectors.

(a) First, just two eigenvectors.

   i. Choose the eigenvectors:

```
choices=[60,80];
```

   ii. Choose their coefficients (these should be between -2 and 2 for the stability of the code):

```
coeffs=[1,-0.25];
```

   iii. What is the linear combination we are about to plot?

   iv. Plot and watch the diffusion of these by choosing a maximum number of time steps `MaxTime` and running the function `DiffuseLinearCombination`:

```
MaxTime=50;
DiffuseLinearCombination(V(:,choices),L(choices),coeffs,MaxTime);
```

   v. Do this again by changing the command lines starting as

```
choices=...
coeffs=...
```

   You can change `MaxTime` if you want also, but making it larger than 500 might make you sit for a very long time. You can also change the pause time between frames by giving a fifth input to function `DiffuseLinearCombination` which is the pause time in seconds.

   vi. Make some observations about the diffusion of a linear combinations of two eigenvectors. Try various linear combinations as needed.

(b) Now, 5 eigenvectors (the code is not set up to do more).

   i. Choose the eigenvectors:

```
choices=[60,70,80,90,100];
```

   ii. Choose their coefficients (these should be between -2 and 2 for the stability of the code):

```
coeffs=[1,-1,1,-1,1];
```

   iii. What is the linear combination we are about to plot?

   iv. Plot and watch (you may want to watch this multiple times) the diffusion of these by choosing a maximum number of time steps `MaxTime` and running the function `DiffuseLinearCombination`:

```
MaxTime=100;
```

```
DiffuseLinearCombination(V(:,choices),L(choices),coeffs,MaxTime);
```

v. Do this again by changing the command lines starting as

```
choices=...
coeffs=...
```

vi. Make some observations about the diffusion of linear combinations of eigenvectors. Try various linear combinations as needed.

(c) Use the above explorations to make a statement about diffusion details for an arbitrary heat state $u = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_m\beta_m$, where the β_i are eigenvectors.