Homework 20

There are closed forms for some finite sums that are "relatively simple" to come by. For example,

$$\sum_{k=1}^{n} 1 = n, \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

We get the first one by noticing that we are just adding the number 1 n times. So we get $n \cdot 1 = n$.

The second we get by rewriting:

$$\begin{aligned}\sum_{k=1}^{n} k &= 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n \\ &= (1+n) + (2 + (n-1)) + (3 + (n-2)) + \ldots \\ &= (n+1) + (n+1) + (n+1) + \ldots\end{aligned}$$

In the above, we pair up the first half of the numbers with the second half of the numbers in the list. So, there are $\frac{n}{2}$ such pairs and they all add to $n + 1$. So

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

The third sum can be obtained with some trickery...be patient at first.

$$\sum_{k=1}^{n} k^3 = \sum_{k=0}^{n} k^3 = \sum_{k=0}^{n} (k+1)^3 - (n+1)^3.$$

Let me stop and tell you that we are getting there and that this last step seems magical, but because the $k = 0$ term is 0, we really do start with 1 on both sums. The second step is just doing what I will do in the following example:

$$0^3 + 1^3 + 2^3 + 3^3 = 1^3 + 2^3 + 3^3 + 4^3 - 4^3.$$

See how the 0 at the beginning is ignored and there's this extra term added at the end that we just subtract again. This seems like a bad idea, but watch what happens:

$$\begin{aligned}\sum_{k=1}^{n} k^3 = \sum_{k=0}^{n} k^3 = \sum_{k=0}^{n} (k+1)^3 - (n+1)^3 &= \sum_{k=0}^{n} (k^3 + 3k^2 + 3k + 1) - (n+1)^3 \\ &= \sum_{k=0}^{n} k^3 + 3\sum_{k=0}^{n} k^2 + 3\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 - (n+1)^3\end{aligned}$$

So now we have this equation:

$$\sum_{k=0}^{n} k^3 = \sum_{k=0}^{n} k^3 + 3\sum_{k=0}^{n} k^2 + 3\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 - (n+1)^3.$$

We can solve for $\sum_{k=0}^{n} k^2$ and we get:

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \frac{1}{3}(n+1)^3 - \sum_{k=1}^{n} k - \frac{1}{3}\sum_{k=0}^{n} 1 \\ &= \frac{1}{3}(n+1)^3 - \frac{n(n+1)}{2} - \frac{1}{3}(n+1) = \frac{2(n+1)^3 - 3n(n+1) - 2(n+1)}{6} \\ &= \frac{(n+1)(2(n^2+2n+1) - 3n - 2)}{6} = \frac{(n+1)(2n^2+n)}{6} = \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

Ok, so now you see why I said that there was some trickery and why I said "relatively simple." Anyway, I want you to use these formulas to calculate the following sums:

1. $\displaystyle\sum_{k=1}^{n}(k^2 - 5k + 2)$

2. $\displaystyle\sum_{k=1}^{n}(k+1)(3k-2)$

3. $\displaystyle\sum_{k=1}^{n}(3k^2 + 4k - 2)$

4. $\displaystyle\sum_{k=1}^{n}(4k^2 - 1)$