LAB #8: FINDING DESCENT TO MINIMIZE

Recall, we want to minimize

$$F(u) = \int_\Omega |\nabla u|^p \, dx + \lambda \int_\Omega |f(x) - u(x)|^q \, dx.$$

Let's first put together some thoughts about minimizing:

(1) Given a function, $f : \mathbb{R}^n \to \mathbb{R}$, at a point $a \in \mathbb{R}^n$, in which direction should you travel to find greatest descent?

(2) Suppose you follow this descent direction, why would you need to go a short distance and then reevaluate the descent direction? What could happen if you took a large step along this direction?

(3) Draw an example of a function where the scenario you described above would happen.

(4) In optimization terminology, looking along a line parallel to a descent direction to a "better" point is called a " line search." As we said in the last problem, if we take too big of a step along this line, we may have a problem. Because we know the direction in which we step is a direction of descent, what can we do to be sure we reach a better point?

(5) Following the steps below, we get a sequence (x_n) of better points.
   (a) At a current point, x_k find the direction of descent d_k (described above)
   (b) Use a line search to find a better point, one where f has a lower value.
   (c) Name the better point x^*.
   (d) Repeat for $x_k = x^*$.

   How can this plan go wrong?

(6) List two ideas to combat the problems that might happen with the above algorithm. Describe how these ideas combat the issues in 5.

(7) When looking for a minimizer of a function, what must be true about the gradient at that point?

(8) We found that the minimizer, $\bar{u}$ of F must satisfy the equation

$$p\nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right) - \lambda q|f - u|^{q-2}(f - u) = 0.$$

   How did we go about getting this equation?

(9) Using the last two questions, what can we use as the "gradient" of $F(u)$?

(10) Use your answers above to determine a direction of fastest descent at a point u.