LAB #7: CHAIN RULE FOR DIRECTIONAL DERIVATIVES

1. Back to $f : \mathbb{R}^n \to \mathbb{R}$

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, in Lab #6 we put together a two step scheme for finding the directional derivative of f in the u direction at a point a. Our scheme went like this:

A. Define a new function $g : \mathbb{R} \to \mathbb{R}$ as $g(t) = f(a + ut)$. That is, we define g along the line going through a and parallel to the vector u.

B. Find $g'(0)$.

(1) Notice that g is defined as $f \circ \ell$, the composition of two functions. One function is $f : \mathbb{R}^n \to \mathbb{R}$. What is ℓ? Be specific.

(2) We want to then explore how to take derivatives of such functions in a more symbolic or formulaic way (rather than just use the limit definition. Explore this with the following functions. That is, find $g'(0)$ for each of the following functions.
   - (a) $f(x, y, z) = x^2 + 2\sqrt{y} + \frac{1}{z}$.
   - (b) $f(x, y, z) = xyz$
   - (c) $f(x, y, z) = \frac{x}{y} + z^2$
   - (d) $f(x, y, z) = r(x) + h(y) + k(z)$, where $g, h, k : \mathbb{R} \to \mathbb{R}$.

(3) In the case when $n = 1$ we would see that the chain rule from Calculus 1: $g'(t) = f'(\ell(t)) \cdot \ell'(t)$. But, f' doesn't make sense here. So, let's extend the Calculus 1 chain rule to the above cases. Write a rule for

$$\frac{d}{dt} f(\ell(t)).$$

2. On to Functionals

(4) Now we want to apply your rule to functionals. First, try to see how it applies in a., b., and c. below. You will want to formulate this rule to use it on d., e., and f. Compute $\frac{d}{dt} g(t)$, where $g(t) = F(u + tv)$ for some u, v functions in the "right" space, for the following.
   - (a) $F : C(\Omega) \to \mathbb{R}$ defined by $F(u) = u(0)$. ($\Omega \subset \mathbb{R}$.)
   - (b) $F : C(0, 1) \to \mathbb{R}$ defined by $F(u) = \int_0^1 u(x)\, dx$.
   - (c) $F : C^1(0, 1) \to \mathbb{R}$ defined by $F(u) = \int_0^1 u'(x)\, dx$.
   - (d) $F : C^1(\Omega) \to \mathbb{R}$ defined by $F(u) = \int_\Omega |\nabla u(x)|\, dx$. ($\Omega \subset \mathbb{R}^2$.)
   - (e) $F : C^1(\Omega) \to \mathbb{R}$ defined by $F(u) = \int_\Omega |\nabla u(x)|\, dx + \int_\Omega u(x)\, dx$ ($\Omega \subset \mathbb{R}^2$.)
   - (f) Now, we write a general formula: Let $L : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. We define $F : C^1(0, 1) \to \mathbb{R}$ by $F(u) = \int_\Omega L(u, \nabla u)\, dx$