Lab #6: Combining Directional Derivatives and Functionals

1. Directional derivative

(1) In the last lab, we found in the case $n > 1, m = 1$ that the derivative map is just the gradient of f. In your own words, describe how the gradient is a map. (Hint: This is question is just asking you to recall what we did in Lab 5.)

(2) Now, let $u \in \mathbb{R}^n$ have length 1, we can define the directional derivative at a point $a \in \mathbb{R}^n$ as

$$D_u(f(a)) = \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t}.$$

Describe with a picture and words what is going on in this limit. (Note: This is a picture we have already drawn in this class.)

(3) Use this definition to find the directional derivative of $f(x, y, z) = x^2 + 2y - z$ at $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ in the direction $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ with $|u| = 1$.

(4) Pick a point a and a direction u and use your computation to find the $D_u(f(a))$ for your choices. What does this derivative mean?

2. How does this tie into our discussion of functionals?

Recall that our goal is to find a way to minimize the functional.

$$F(u) = \int_\Omega |\nabla u(x)|^p \, dx + \lambda \int_\Omega |f(x) - u(x)|^q \, dx.$$

The next set of questions will guide you through some preliminary steps by first considering minimization in some simpler settings.

(1) Using your knowledge of 1-dimensional derivatives, write the procedure one usually takes to find a minimum of a function.

(2) Now, suppose we have a multivariable function $f : \mathbb{R}^n \to \mathbb{R}$. How would we find the minimum of f along a specified line?

(3) We'd like to extend these ideas to find the minimum of a functional $F : \mathcal{C}^\infty(\Omega) \to \mathbb{R}$, where $\Omega \subset \mathbb{R}$ is an open domain space, along a specified line. Towards this end, how would you describe a "line path" in the domain (function space) $\mathcal{C}^\infty(\Omega)$?

(4) Using the definition of directional derivatives of functions from the first part of this lab, give a definition for the directional derivative of a functional $F : \mathcal{C}^\infty(\Omega) \to \mathbb{R}$.

(5) Test out your definition on some functionals $F : \mathcal{C}^\infty(\Omega) \to \mathbb{R}$. Find the "directional derivative" for the functionals below in the "direction" of some fixed function $v \in \mathcal{C}^\infty(\Omega)$.

   (a) $F(u) = \displaystyle\int_0^1 u(x) \, dx.$

   (b) Let $f(x)$ be a fixed function in $\mathcal{C}^\infty$ and define $F(u) = \displaystyle\int_0^1 u(x) - f(x) \, dx.$

   (c) $F(u) = \displaystyle\int_0^1 |\nabla u(x)| \, dx.$ (Be careful here to recognize what the $|...|$ are representing.)

   (d) For a functional $F : \mathcal{C}^\infty(\Omega) \to \mathbb{R}$ and the function v given by $v(x) = x$, what is the directional derivative of F in the direction v?