Lab #5: Linear Maps

In this lab, we want to explore the following definition:

Definition 1. *Suppose A is an open set in $\mathbb{R}^n$, $f : A \to \mathbb{R}^m$, and $a \in A$. We say that f is differentiable at $a \in \mathbb{R}^n$ if $\exists L : \mathbb{R}^n \to \mathbb{R}^m$ so that L is linear and*

$$\lim_{|h| \to 0} \frac{|f(a+h) - f(a) - Lh|}{|h|} = 0.$$

1. **Case:** $n = m = 1$

(1) Rewrite this definition when $n = m = 1$.
(2) What is L?
(3) What does Lh mean?
(4) Find L for the function $f(x) = x^2$ at $a \in \mathbb{R}$. Prove your result.

2. **Case:** $n > 1, m = 1$

(1) Think about the definition when $n > 1, m = 1$ and rewrite it when $n = 2, m = 1$.
(2) What is L?
(3) What does Lh mean in this setting?
(4) What do we mean when we use the $|\dots|$ symbols?
(5) Find L for the function $f(x, y, z) = x^2 + 2y - z$ at $a \in \mathbb{R}^3$. Prove your result.
(6) Find L for the function $f(x, y) = xy$ at $a \in \mathbb{R}^2$. Prove your result. (Homework)

3. **Case:** $n > 1, m > 1$

(1) What is L?
(2) What does Lh mean in this setting?
(3) What do we mean when we use the $|\dots|$ symbols?
(4) Find L for the function $f(x, y, z) = \begin{pmatrix} x^2 + 2y - z \\ xy \end{pmatrix}$ at $a \in \mathbb{R}^3$. No need to prove your result (it's tedious).

4. More practice and questions

(1) When considering our denoising problem, how do you think derivatives will play a role?
(2) Use the above to find L in each of the following situations. Be sure to carefully (rigorously) define L.
   (a) $f(x, y, z) = |(x, y, z)|$ (Be careful to recognize what we mean by the $|\dots|$ here).
   (b) $f(x, y) = \begin{pmatrix} y \\ \\ x^2 \end{pmatrix}$