Lab #4: Functionals

First, we define a functional.

Definition 1. *Let $\mathcal{F}$ be a function space. A functional $F : \mathcal{F} \to \mathbb{R}$ is a mapping that takes in as inputs functions and outputs real numbers.*

Definition 2. *We say that F is a linear functional if, as a map, F is linear. That is, F satisfies*

$$\forall u, v \in \mathcal{F}, \forall \alpha, \beta \in \mathbb{R} \quad F(\alpha u + \beta v) = \alpha F(u) + \beta F(v).$$

1. Creating functionals

(1) Let $\mathcal{F} = \mathcal{C}([0,1])$

(a) Give an example of a functional $F : \mathcal{F} \to \mathbb{R}$.

(b) Is the functional you defined linear? If so, find a functional that is not linear. If not, find a linear functional.

(c) For your linear functional, prove that it is linear.

(2) Let $\mathcal{F} = \mathcal{C}^1([0,1])$

(a) Give an example of a functional $F : \mathcal{F}([0,1]) \to \mathbb{R}$. Choose a functional that is not the same as either of the functionals you found above.

(b) Is the functional you defined linear? If so, find a functional that is not linear. If not, find a linear functional.

(c) For your linear functional, prove that it is linear.

(3) Let $\mathcal{F} = \mathcal{C}^\infty([0,1])$

(a) Give an example of a functional $F : \mathcal{F}([0,1]) \to \mathbb{R}$. Choose a functional that is not the same as either of the functionals you found above.

(b) Is the functional you defined linear? If so, find a functional that is not linear. If not, find a linear functional.

(c) For your linear functional, prove that it is linear.

2. **Analysis of Functionals**

Our discussion on functions in this class began with continuity. Let's see if we can extend the idea to functionals.

(1.) Let's recall what we know about functions.

(a.) Write the definition of continuity for a function $f : A \to \mathbb{R}$.

(b.) How do metrics get involved in the definition of continuity?

(c.) Not all functions were continuous . Sometimes it was good enough to have upper semicontinuous or lower semicontinuous. Write the definition of lower semicontinuous.

(d.) How do metrics get involved in the definition of lower semicontinuous.

(e.) Suppose we have a metric space of functions, $(\mathcal{F}, d)$, with metric d. Use this to extend the definitions of continuity and lower semicontinuity to functionals.

(2.) So, we need a metric space of functions. Write the definition of a metric d and a metric space (X, d).

(3.) Now, we will create some metric spaces of functions.

(a.) Let's consider first, $X_1 = C([0,1])$. Give an example of a metric that measures the difference between two functions in X_1.

(b.) Now consider, $X_2 = C^1([0,1])$. Give an example of a metric that measures the difference between two functions in X_2. Be sure to choose a metric that would work in X_2 but not X_1.

(c.) Consider, $X_3 = C^k([0,1])$ for some $k \in \mathbb{N} \setminus \{1\}$. Give an example of a metric that measures the difference between two functions in X_3. Choose a metric that would work in X_3 but not X_1 nor X_2.

(d.) Finally consider, $X_4 = C^\infty([0,1])$. Discuss how you would extend a metric like the metric you found in X_3 to work in X_4.

(4.) For each of your metrics above, calculate $d(f,g)$ where $f(x) = x^2$ and $g(x) = x^3$.

(5.) How can you adjust each of your metrics above to be functionals on the corresponding spaces?