MATH 352: HOMEWORK 1
DUE TUESDAY JANUARY 26

This homework serves as a review of past classes (including Math 351).

(1) Find $\lim_{x\to 5} \dfrac{(x-5)^2-25}{x-5}$. Prove your answer using the definition for the limit of a function.

(2) On your final exam last semester, we defined the "greater" limit and the "lesser" limit. In actuality, these are called the limit supremum and the limit infimum (written $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$). Here are the definitions again:

Let $(z_n) : \mathbb{N} \to \mathbb{R}$ be a bounded sequence.
We define the <u>*limit supremum*</u> of (z_n) by

$$\limsup_{n\to\infty} z_n = \lim_{n\to\infty} g_n, \text{ where } g_n = \sup\{z_k : k \geq n\}.$$

Similarly, we define the <u>*limit infimum*</u> of (z_n) by

$$\liminf_{n\to\infty} z_n = \lim_{n\to\infty} \ell_n, \text{ where } \ell_n = \inf\{z_k : k \geq n\}.$$

Draw a picture of a sequence (a_n) that does not converge (plot in $\mathbb{R}^2$ with the horizontal axis representing n and the vertical axis representing a_n). Draw the continuous functions (each in a different color) defined by

$$f(x) = (g_n - g_{n-1})(x-n) + g_n, \text{ for } n-1 < x \leq n. \text{ And,} h(x) = (\ell_n - \ell_{n-1})(x-n) + \ell_n, \text{ for } n-1 < x \leq n.$$

How can you find the $\liminf_{n\to\infty}(a_n)$ and $\limsup_{n\to\infty}(a_n)$ from this picture?

(3) Linear Algebra review:

(a) Show that the space of continuous functions $\mathcal{C}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} | \ f \text{ is continous}\}$ is a vector space with scalars taken from $\mathbb{R}$. Here, we define addition and scalar multiplication to be pointwise:

The function $f+g$ is defined by $(f+g)(x) = f(x) + g(x)$
and
The function (αf) is defined by $(\alpha f)(x) = \alpha(f(x))$.

Recall the 10 properties of a vector space V are:

(i) V is closed under scalar multiplication.
(ii) V is closed under addition.
(iii) Addition is commutative.
(iv) Addition is associative.
(v) Scalar multiplication is associative.
(vi) Scalar multiplication distributes over scalar addition.
(vii) Scalar multiplication distributes over vector addition.
(viii) There exists a zero vector in V.
(ix) There exist additive inverses in V.
(x) There exists a scalar identity.

(b) In linear algebra, we learn that if V and W are vector spaces and if a transformation $T : V \to W$ satisfies $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$ then T is called linear. Show that $L : \mathcal{C}(\mathbb{R}) \to \mathbb{R}$ defined by $L(f) = \lim_{x \to 3} f(x)$ is linear.

(4) Given the function $f(x, y) = 2x^2 + xy - y^2 + x$, find
  (a) The derivative of f in the direction $v = \langle 1, 2 \rangle$ at the point $(1, 1)$.
  (b) The direction of steepest ascent at the point $(1, 3)$. Describe how you know this is the direction of steepest ascent.
  (c) All extrema.
  (d) Find all extrema of f on the ellipse $x^2 + 3y^2 = 4$.
  (e) Show through explanations and pictures (not proof) that the directional derivative $D_v f(p)$ at the point p in the direction v is given by

$$D_v f(p) = \nabla f(p) \cdot \frac{v}{||v||}.$$