Introduction to Metric Spaces

A *metric space* is a set X where we have some way of measuring the distance between two points.

Exercise 1: Let's look at a few ideas before being more specific about this. Do each of the following.

a. X = the set of all students in Analysis. Come up with an idea for a way to measure the distance between the students. You really don't need to know the answer to the distance between student x and student y, just brainstorm ideas.

b. Choose a person at your table that has some pictures on their phone that they are willing to share. (If none of you want to share pictures, find a set of class appropriate pictures online to do this.) Determine a way to measure distance between these pictures.

c. Everybody at your table write the alphabet in front of you in your regular handwriting. Discuss a way to measure the distance between your handwriting and those at your table. (You can choose just one letter if it's easier to think about).

We really want to know if any of the above sets (with the distance that you made up) can be considered a metric space. We want a rigorous definition of distance so that we can discuss the mathematics of specific metric spaces related to some cutting-edge applications. Here is the definition:

Definition[**metric**] Given a set X, a function $d : X \times X \to \mathbb{R}$ is a *metric* (or distance function) on X if for all $x, y \in X$ we have the following three properties:

a. Distances are positive except when between a element and itself:

$$d(x, y) \geq 0 \quad \text{with } d(x, y) = 0 \quad \iff \quad x = y.$$

b. Distance is symmetric:

$$d(x, y) = d(y, x).$$

c. The shortest distance from one point to another does not require going through a third point (ie: the triangle inequality):

$$\text{for all } z \in X \quad d(x, z) + d(z, y) \leq d(x, y).$$

Definition[**metric space**] A set X with a distance function d is called a *metric space.*

Exercise 2: Now go back to your examples above and determine which can be considered a metric space. If your metric doesn't follow all three properties, find a way to adjust it (or invent a metric) so that you have a metric space. Justify your results.

Exercise 3: Name 2 metric spaces that you know already. (Remember, a set is not a metric space. A set with a distance function is a metric space.) Prove that your examples are indeed metric spaces.

Examples of metric spaces

Here are some examples of metric spaces.

<u>**Exercise 4**</u> (The trivial metric, or discrete metric.) Let X be a set, and $d : X \times X \to \mathbb{R}$ be the function that maps $d(x, y) = 1$ if $x \neq y$, and $d(x, x) = 0$.

a. Prove that this is a metric space.

b. Why does it deserve the titles "trivial" and "discrete?"

<u>**Exercise 5**</u> $\mathbb{R}$ with the metric $d(x, y) = |x - y|$. Prove this is a metric.
We can measure the distance between two points in $\mathbb{R}^2$ in a number of different ways.

<u>**Exercise 6**</u> **The "Euclidean Metric"** $\mathbb{R}^2$ with the standard distance formula:

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

<u>**Exercise 7**</u> **The "Taxicab" Metric.** $\mathbb{R}^2$ with the metric

$$d_1((x_1, x_2), (y_1, y_2)) := |x_1 - y_1| + |x_2 - y_2|.$$

a. Prove that $(\mathbb{R}^2, d_1)$ is a metric space.

b. Explain why it makes sense to call the distance function d_1 the "taxicab" metric on $\mathbb{R}^2$.
<u>**Exercise 8**</u> How does the taxicab metric above differs from the usual ("Euclidean") distance on $\mathbb{R}^2$?

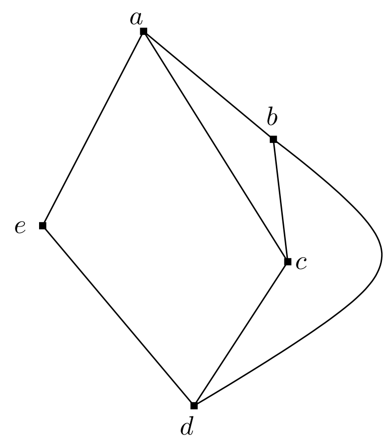
<u>**Exercise 9**</u> Besides the Euclidean distance and the taxicab metric, here is a third way we might measure the distance between two points in $\mathbb{R}^2$.

$$d_\infty((x_1, x_2), (y_1, y_2)) := \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

Prove this is also a metric.

<u>Note</u>: All of these metrics can be easily generalized to higher dimensions.

<u>Exercise 10</u> Let G be a *graph*: a finite set V of vertices with a set of edges E. (Note that not all possible edges need to be included in a graph.) Suppose that G is connected (i.e., there exists a path of edges between any two vertices). Define the distance $d(x, y)$ between two vertices x and y to be the number of edges in the shortest path connecting the x and y. Then (V, d) is a metric space.



a. For the 5 point metric space depicted above, find the following distances: $d(c, a)$, $d(a, d)$, and $d(c, d)$.

b. Now consider a general graph V. Explain why (V, d) is a metric space.

c. Explain why we need the graph to be connected.

<u>Remark:</u> It turns out that there are (fairly simple) graph metric spaces that cannot be *embedded* without distortion into $\mathbb{R}^n$ for any n!

<u>Another Example:</u> A weirder example of a metric space is the unit disc in $\mathbb{C}$ with the so-called *hyperbolic metric* $d(z, w) := 2\tanh^{-1}\left|\frac{z-w}{1-z\overline{w}}\right|$. (Proving that this is a metric is **extra credit**.)

Another Remark: Many objects of practical importance (soundwaves, images, DNA, abstract datasets) can be given structure with distance functions. One can apply metric space analysis to these spaces to perform tasks such as data compression and facial or voice recognition.