MATH 351 – AXIOMS OF $\mathbb{R}$

We will assume the following as axioms of the real number system. This is not a standard list. Typically, we want axioms to be desirable and necessary. They should be desirable in the sense that they are facts that we want to be true. They should be necessary in the sense that all of our desired facts about the real numbers follow from them, and not from any smaller, simpler set of axioms. You may want to think about creating a smaller set of axioms by eliminating some of those given here. Here are the facts we will assume about the set $\mathbb{R}$.

1. (The Field Axioms) There are two binary operations on $\mathbb{R}$, $+$ and $\cdot$, with the following properties:

(1) $+$ and $\cdot$ are commutative operations.
(2) $+$ and $\cdot$ are associative operations.
(3) There exists a real number (called 0) such that for all $x \in \mathbb{R}$, $x+0 = x$.
(4) There exists a real number distinct from 0 (called 1), such that for all $x \in \mathbb{R}$, $x \cdot 1 = x$.
(5) For all $x \in \mathbb{R}$, there exists a real number (called $(-x)$ such that $x + (-x) = 0$.
(6) For all $x \in \mathbb{R}$ not equal to 0, there exists a real number (called $\frac{1}{x}$ or x^{-1}) such that $x \cdot \frac{1}{x} = 1$.
(7) For all real numbers x, y, z, $x \cdot (y + z) = x \cdot y + x \cdot z$.

Definitions: $x - y = x + (-y)$. $\frac{x}{y} = x \cdot y^{-1}$. $xy = x \cdot y$.

2. (Uniqueness of Inverses) If $x \in \mathbb{R}$, there is only one real number y such that $x + y = 0$. If $x \neq 0$, there is only one $y \in \mathbb{R}$ such that $x \cdot y = 1$.

3. $(-0) = 0$.

4. If $x \in \mathbb{R}$, then $-(-x) = x$.

5. For all $x \in \mathbb{R}$, $x \cdot 0 = 0$.

6. If $x, y \in \mathbb{R}$, $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$.

7. If x and y are non-zero, then

(1) $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.
(2) $\frac{x}{y} \neq 0$ and $\frac{1}{\frac{x}{y}} = \frac{y}{x}$.

8. For all $x \in \mathbb{R}$, $(-x) = (-1) \cdot x$.

9. If $x, y \in \mathbb{R}$ and $x \cdot y = 0$, then $x = 0$ or $y = 0$.

10. There exists a subset $\mathbb{R}^+$ of $\mathbb{R}$, called the set of positive numbers, which satisfies

(1) If $x, y \in \mathbb{R}^+$, then $x + y \in \mathbb{R}^+$ and $x \cdot y \in \mathbb{R}^+$.
(2) If $x \in \mathbb{R}$, then one and only one of these is true:
  - $x = 0$.
  - x is positive.

- $(-x)$ is positive.

Definitions: The complement $\mathbb{R} \setminus (\mathbb{R}^+ \cup \{0\})$ is the set of negative numbers $\mathbb{R}^-$.
If $x, y \in \mathbb{R}$, we say $x \leq y$ if and only if $y - x \in \mathbb{R}^+ \cup \{0\}$. $\geq$, $<$, and $>$ are defined similarly.

11. 1 is a positive number.

12. $\forall x \neq 0$, $x \neq (-x)$.

13. ($(\mathbb{R}, \leq)$ is a partially ordered set) The relation $\leq$ satisfies
(1) $\forall x \in \mathbb{R}$, $x \leq x$.
(2) $\forall x, y \in \mathbb{R}$, if $x \leq y$ and $y \leq x$, then $x = y$.
(3) $\forall x, y, z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

14. ($(\mathbb{R}, \leq)$ is a totally ordered) If $x, y \in \mathbb{R}$, then one and only one of the following is true:

- $x < y$.
- $x = y$.
- $x > y$.

15. Let $a, b, c, d \in \mathbb{R}$.
(1) If $a > b$ and $c \geq d$, then $a + c > b + d$.
(2) If $a > b > 0$ and $c \geq d > 0$, then $ac > bd$.
(3) If $a > b$ and $c < 0$, then $ac < bc$.

16. Let $a, b, c \in \mathbb{R}$.
(1) If $a \geq 1$, then $a^2 \geq a$.
(2) If $0 < a < 1$, then $a^2 < a$.
(3) If $0 < a < b$, then $a^2 < b^2$.
(4) If a and b are nonzero and have the same sign, and $a \leq b$, then $\frac{1}{a} \geq \frac{1}{b}$.
(5) $a > 1$ if and only if $0 < \frac{1}{a} < 1$.
(6) If $a \leq c$ and $b > 0$, then $\frac{a}{b} \leq \frac{c}{b}$.
(7) If $a \leq c$ and $b < 0$, then $\frac{a}{b} \geq \frac{c}{b}$.

17. There exists an injective map $f : \mathbb{N} \to \mathbb{R}$ such that $\forall n, m \in \mathbb{N}$, $f(n+m) = f(n) + f(m)$ and $f(n \cdot m) = f(n) \cdot f(m)$. Thus, we can say $\mathbb{N} \subset \mathbb{R}$ and $\mathbb{R}$ is infinite.

18. There exists an injective map $f : \mathbb{Q} \to \mathbb{R}$ such that $\forall x, y \in \mathbb{Q}$, $f(x + y) = f(x) + f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$. Thus, we can say $\mathbb{Q} \subset \mathbb{R}$.

19. $\frac{1}{2} < 1$.

20. If $k \in \mathbb{N}$ and $a \in \mathbb{R}$, then $k \cdot a = a + a + \cdots + a$ (k times).

Definitions: Let $S \subset \mathbb{R}$ and $x \in \mathbb{R}$. x is called an upper bound for S if $\forall y \in S$, $x \geq y$. x is called the least upper bound, or supremum of S if it is an upper bound for S and if $x \leq u$ for all upper bounds u of S. S is called "bounded above" if there exists an upper bound for S.

21. Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound. (Note: this does not say that the least upper bound is an element of the set).