MATH 351 Fall 2015 Homework 9 Due: Tuesday 11/17

Read Chapter 2 sections 4,5. For each of the following problems, use the metric space $(\mathbb{R}, d)$ where d is the usual metric. Below JFF means "Just For Fun," so you can ignore it if you don't want to be challenged by it. I will look over and give feedback for any that are turned in, but this problem will not affect your grade.

(1) Show that

$$\text{if } \sum_{n=0}^{\infty} 2^n b_{2^n} \text{ diverges} \quad \text{then} \quad \sum_{n=1}^{\infty} b_n \text{ also diverges.}$$

(2) Let $(a_n) : \mathbb{N} \to \mathbb{R}$ be the sequence given by

$$a_n = 2^{r_n}, \quad \text{where} \quad r_n = \frac{2^n - 1}{2^n}.$$

Determine whether (a_n) converges or diverges. Prove your answer.

(3) Compute and prove your answer

$$\lim_{n\to\infty} \frac{2n^4 + 8}{n^5 - 6}.$$

(4) Suppose $(a_n) : \mathbb{N} \to \mathbb{R}$ converges to a.
  (a) Prove: If $a > 0$ then $\exists N \in \mathbb{N}$ so that $a_n > 0$ for every $n \geq N$.
  (b) Prove or disprove: If $a_n < 0$ for every tenth natural number n, then $a < 0$.
  (c) Generalize the above two statements (don't prove them).

(5) Give a sequence that does not converge, but has at least two convergent subsequences.

(6) Prove or disprove: Let $(a_{n_j}) : \mathbb{N} \to \mathbb{R}$ be convergent subsequence of the bounded sequence $(a_n) : \mathbb{N} \to \mathbb{R}$. Then $(a_n) : \mathbb{N} \to \mathbb{R}$ converges.

(7) Prove or disprove: Given a convergent sequence, then every subsequence also converges to the same limit.

(8) Give an example (if such a sequence exists) of a sequence whose terms are never 1 or -1, but contains at least one subsequence that converges to 1 and one subsequence that converges to -1. If such a sequence does not exist, it is sufficient to state so without proof, but be sure you are correct.

(9) Write, in your own words a rigorous proof of the Bolzano Weierstrauss theorem. You may use the proof in the book, but you need rewrite it in your own words.

(JFF) Define a sequence whose range is $\mathbb{Q}$ or show that such a sequence doesn't exist.