MATH 351 Fall 2015 Homework 7 Due: Tuesday 10/22

Read Chapter 2 sections 1-3.

(1) Let X be the given metric space with the given metric d. Determine whether or not the sequence $(a_n) : \mathbb{N} \to X$ (where a_n is given below) converges. Prove or find a counterexample.
 (a) $a_n = \frac{2n+3}{n+4}$ and d is the standard metric
 (b) $a_n = n$ and $d(x,y) = \left| \frac{1}{|x+2|} - \frac{1}{|y+2|} \right|$
 (c) $a_n = 1$ and d is the trivial metric

(2) Prove or disprove: Let $\mathbb{R}$ be the metric space with the standard metric $d(x,y) = |x-y|$. The sequence $\left(\sin\left(\frac{1}{n}\right)\right) : \mathbb{N} \to \mathbb{R}$ converges.

(3) Prove or disprove: Let $\mathbb{R}$ be the metric space with the standard metric. The sequence $(a_n) : \mathbb{N} \to \mathbb{R}^m$ converges where

$$a_n = \left(\frac{1}{n}, \frac{1}{2n} + \frac{1}{n}, \frac{1}{3n} + \frac{1}{2n} + \frac{1}{n}, \ldots, \sum_{k=1}^{m} \frac{1}{kn} \right).$$

(4) Prove and generalize: Let X be a metric space with metric d_X and Y be a metric space with metric d_Y. Suppose

$$(a_n) : \mathbb{N} \to X \quad \text{and} \quad (b_n) : \mathbb{N} \to Y$$

both converge. Then

$$((a_n, b_n)) : \mathbb{N} \to X \times Y$$

converges, where

$$X \times Y = \{(x,y) | x \in X, y \in Y\}$$

is the metric space with metric

$$d((a_1, a_2), (b_1, b_2)) = d_X(a_1, b_1) + d_Y(a_2, b_2).$$

(5) Prove or disprove: Let X be a metric space with metric d. Suppose the two sequences $(a_n), (b_n) : \mathbb{N} \to X$ both converge. Let

$$c_n = \begin{cases} a_n & \text{if } n \text{ is even} \\ b_n & \text{if } n \text{ is odd} \end{cases}$$

then the sequence $(c_n) : \mathbb{N} \to X$ also converges if and only if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

(6) Let $\mathbb{R}$ be a metric space with metric d. Suppose that the sequence $(a_n) : \mathbb{N} \to \mathbb{R}$ converges in this metric space. Let

$$b_n = \frac{1}{n} \sum_{k=1}^{n} a_k.$$

The statement:

$(b_n) : \mathbb{N} \to \mathbb{R}$ also converges in the same metric space to the same limit. That is,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

is difficult to prove. Instead
 (a) Find a metric that this is not true for.
 (b) Show it is true for the standard metric on $\mathbb{R}$.