MATH 351 – FOM HOMEWORK

1. Solutions

A. **Statement: $\sqrt{2}$ is irrational.**
proof: Assume, for the sake of contradiction, that $\sqrt{2} \in \mathbb{Q}$. Then $\exists x, y \in \mathbb{Z}, y \neq 0$, and $gcd(x, y) = 1$ so that $\sqrt{2} = \frac{x}{y}$. Then $2 = \frac{x^2}{y^2}$. This means that $x^2 = 2y^2$. So x^2 is even and therefore x is even. That is, $x = 2k$ for some $k \in \mathbb{Z}$. Then $y^2 = 2k^2$ and so y is also even, a contradiction. Thus $\sqrt{2}$ is irrational. $\square$

(1) **Prove or disprove:**
   (a) $(1,3)$ **is a subset of** $(2,4)$.
   Statement: $(1,3)$ is <u>not</u> a subset of $(2,4)$.
   Counterexample: Notice that $2 \in (1,3)$, but $2 \notin (2,4)$. Thus $(1,3) \not\subset (2,4)$.
   (b) $(2,3)$ **is a subset of** $(1,4)$.
   Statement: $(2,3) \subset (1,4)$.
   proof: Let $a \in (2,3)$. We want to show that $a \in (1,4)$. Notice that
   $$1 < 2 < a < 3 < 4.$$
   Thus, $a \in (1,4)$. So, $(2,3) \subset (1,4)$. $\square$

(2) **Use problem 1 to make a general statement about $a, b, c,$ and d so that**
   $$(a,b) \subset (c,d).$$

   Statement: If for $a, b, c, d \in \mathbb{R}$ we have that $c < a < b < d$, then $(a,b) \subset (c,d)$.

(3) **Prove or disprove the statement "If a is a rational number and b is an irrational number, then $a + b$ is irrational."**
   Statement: If a is a rational number and b is an irrational number, then $a + b$ is irrational.
   proof: Let $a \in \mathbb{Q}$ and $b \in \mathbb{R} \setminus \mathbb{Q}$. And, for the sake of contradiction, assume that $a + b \in \mathbb{Q}$. Then, $\exists x, y, p, q \in \mathbb{Z}$ with $y, q \neq 0$ and $a = \frac{x}{y}$ and $a + b = \frac{p}{q}$. Then we have
   $$\begin{aligned} b &= a + b - a \\ &= \frac{p}{q} - \frac{x}{y} \\ &= \frac{py - qx}{qy} \quad \in \mathbb{Q}. \\ &\rightarrow\leftarrow \end{aligned}$$
   This contradicts the assumption that $b \in \mathbb{R} \setminus \mathbb{Q}$. Thus, $a + b \in \mathbb{R} \setminus \mathbb{Q}$. $\square$

(4) **Consider the Well-Ordering Principle:**
   Any nonempty subset $S \subset \mathbb{N}$ contains a least element.

   (a) **Write this in another way so that the meaning of "contains a least element" is more mathematical.**

Let $S \subset \mathbb{N}$, $S \neq \emptyset$, then $\exists s \in S$ so that $\forall x \in S$ we have that $s \leq x$.

(b) **Write the Principle of Mathematical Induction.**
Let $P(k)$ be a statement that depends on $k \in \mathbb{N}$. If
- $P(1)$ is true.
  And
- If, for $n \in \mathbb{N}$, $P(n)$ is true then $P(n+1)$ is true.

are both true, then $P(k)$ is true $\forall k \in \mathbb{N}$.

(c) **Use the Well-Ordering Principle to prove the Principle of Mathematical Induction.**
Statement: Suppose the Well-Ordering Principle is true. That is, suppose that all nonempty subsets of $\mathbb{N}$ contain a least element. Let $P(k)$ be a statement depending on $k \in \mathbb{N}$. Suppose $P(1)$ is true and whenever $P(n)$ is true that $P(n+1)$ is also true. Then $P(k)$ is true for all $k \in \mathbb{N}$.
proof: Suppose $P(k)$ is a statement depending on $k \in \mathbb{N}$ and $P(1)$ is true. Suppose also that whenever $P(n)$ is true so is $P(n+1)$. We want to show that $P(k)$ is true for all $k \in \mathbb{N}$. Let us now define the set

$$S := \{a \in \mathbb{N} | P(a) \text{ is false}\} \subset \mathbb{N}.$$

Because $S \subset \mathbb{N}$, we know that by the Well-Ordering Principle, S is either empty or S has a smallest element. Suppose $S \neq \emptyset$. Then S must have a least element. Call this element s. By definition of S, $P(s)$ is false. But, then $P(s-1)$ must also be false (for otherwise by our assumption above, $P(s)$ is true). Since $P(s-1)$ is false, $s - 1 \in S$. But $s - 1 < s$ so s cannot be the least element of S. This means that S cannot have a least element. Thus, S cannot be nonempty. That is, $S = \emptyset$. So, $\nexists a \in \mathbb{N}$ so that $P(k)$ is false. That is, $P(k)$ is true $\forall k \in \mathbb{N}$. *qed*

(d) **Use the Principle of Induction to prove the Well-Ordering Principle.**
Statement: Assume that the Principle of Induction is true. If $S \neq \emptyset$, $S \subset \mathbb{N}$ then, $\exists s \in S$ so that $s \leq x \ \forall x \in S$.
proof: Suppose $S \neq \emptyset$, $S \subset \mathbb{N}$. Let us also suppose, for the sake of contradiction, that S does not contain a least element. Now, let us define the statement
P(k): "The set S does not contains the elements $1, 2, \ldots, k$."
Note that we know that since S does not have a least element, $P(1)$ is true. Now assume that $P(n)$ is true for some $n \in \mathbb{N}$. Then $n + 1 \notin S$ for otherwise $n + 1$ would be the smallest element of S. Thus $P(n+1)$ is true. So by the principle of induction, $P(k)$ is true for all $k \in \mathbb{N}$. But this means that $S = \emptyset$, a contradiction. Thus, S must have a least element. That is, the Well-Ordering Principle is true. $\square$

(5) **Translate each statement into English.**
(a) $\forall \, p \in P (q > p)$. **(Here, P is the set of prime numbers.)**
All primes are less than a number q.
(b) $\forall \, v \in \mathbb{R} \ \exists \, u \in \mathbb{R} \ (\frac{1}{u} = v)$.
All real numbers have a reciprocal.
(c) $\forall \, x \in \mathbb{R} \ \ \forall \, y \in \mathbb{R} \ (x^3 - x = y^3 - y \Rightarrow x = y)$. **(Hint: This is really a statement about a particular function.)**
The function $f(x) = x^3 - x$ is one-to-one.
(6) (a) **[i.)] Is the statement in part 5b True or False? Why?**
The statement is false because $0 \in \mathbb{R}$, but 0 doesn't have a reciprocal.

(b) [ii.)]**Is the statement in part 5c True or False? Why?**
The statement is false because $f(0) = f(1)$ and $1 \neq 0$ so f is not 1-1.

(7) **Let $S \subseteq \mathbb{R}$. Consider the following two quantified statements, labelled F and G.**

$$F : \exists z \in S \ \forall x \in S \ (x \leq z) \qquad G : \forall x \in S \ \exists z \in S \ (x \leq z)$$

(a) **Write down the negation of statement F without using the negation symbol.**

$$\neg F : \quad \forall z \in S \quad \exists x \in S(x > z)$$

(b) **Describe in words what statement G means.**
G : Every element of S has an element at least as big as it.

(c) **Are the two statements logically equivalent? If you think they are say why. If you think they aren't, say why and give a set which makes one true and the other false.**
These statements are not logically equivalent. F says that S has a maximum element, namely z. G says that every element has another element bigger than it. So, here is an example of a set that satisfies G and not F:

$$S = \{x \in \mathbb{Q} | x < \pi\}.$$

Notice that since we can approximate π with rational numbers,

$$3, 3.1, 3.14, 3.141, 3.1415, ...$$

We can always find a rational bigger in S that is bigger than another rational number. So G is true. But, F is not because $\pi \notin S$ and so we cannot find an element in S bigger than all others.

(8) **For this problem, we are considering the following statement:**
$$\text{If } A \subseteq B \text{ and } C \subseteq D, \text{ then } A \cap C \subseteq B \cap D.$$

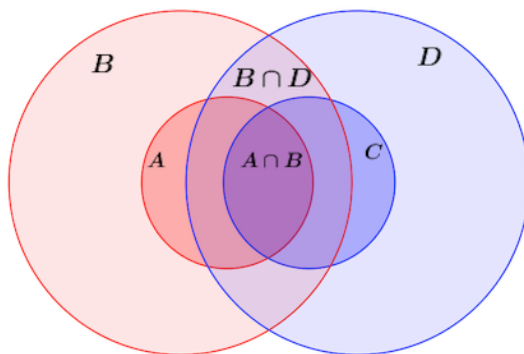(a) **If possible, draw a Venn Diagram that shows this statement.**

FIGURE 1. Venn Diagram

(b) **Give an example of sets $A, B, C,$ and D making this statement true.**
example: Let $A = \{1, 2\}, B = \{1, 2, 3\}, C = \{1, 4\}$ and $D = \{1, 2, 4\}$. Then we have $A \cap C = \{1\} \subseteq \{1, 2\} = B \cap D$ where $A \subseteq B$ and $C \subseteq D$.

3

(c) **Below is a "proof" of this statement. Explain whether it does or does not prove the statement. If you think it does prove the statement, discuss what you would do to improve the proof. If you think it doesn't prove the statement, specifically state *all* the mistakes.**
Proof: Assuming the hypothesis. Take $x \in A$. Then $x \in B$ because $A \subseteq B$. Take $y \in C$. Then $y \in D$ because $C \subseteq D$. So x and y are both in $B \cap D$. Thus $A \cap C \subseteq B \cap D$.
It doesn't prove the statement. Here are the mistakes: the proof should include the statement being proved and state the hypothesis; the strategy is wrong. To prove $X \subset Y$, you must start with $x \in X$ and show that $x \in Y$. Below is a good proof.
statememt: If $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$.
proof: Suppose $A \subseteq B$ and $C \subseteq D$. Let $x \in A \cap C$. We want to show that $x \in B \cap D$. Notice that

$$x \in A \subseteq B \text{ and } x \in C \subseteq D$$

Thus, $x \in B$ and $x \in D$. This implies that $x \in B \cap D$. So, $A \cap C \subseteq B \cap D$. $\square$

(9) **Consider the statement**

"If $x < y$ are elements of $\mathbb{Q}$, then there exists $r > 0$ in $\mathbb{R}$ such that $x + r < y - r$."

(a) **Use quantifiers on the predicate $Q(x, y, r)$:**
$$\text{"} x + r < y - r \text{"}$$
to create an equivalent statement.
$\forall x, y \in \mathbb{Q} \ni x < y \ \exists r \in \mathbb{R} \ni r > 0 \, (x + r < y - r)$

(b) **Prove or disprove this statement.**
Statement: $\forall x, y \in \mathbb{Q} \ni x < y \ \exists r \in \mathbb{R} \ni r > 0 \, (x + r < y - r)$
proof: Suppose that $x, y \in \mathbb{Q}$ such that $x < y$. Choose $r = (y - x)/3 > 0$. We want to show that $x + r < y - r$. Notice that

$$
\begin{aligned}
(x + r) - (y - r) &= \quad x - y + 2r \\
&= \quad x - y + \frac{2y - 2x}{3} \\
&= \quad \frac{x - y}{3} \\
&< \quad 0
\end{aligned}
$$

Thus, we have $x + r < y - r$. $\square$

(10) **Use a truth table to prove that an implication $P \Rightarrow Q$ is logically equivalent to $Q \vee \neg P$.**

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$\sim P$	$\sim P \lor Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

FIGURE 2. Truth Table